

Analysis and Prediction of the Debt Scale and Interest Rate Trends in Japan and Canada by Differential Equations

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Abstract. Advanced economies such as Japan and Canada have experienced long run fiscal deficits amid population ageing and public health crises, yet their debt to Gross Domestic Product (GDP) ratios follow markedly different growth paths. This paper builds a first order linear differential equation debt model incorporating the Bohn fiscal feedback rule, and adopts the integral factor method to explore debt evolution divergence and interest rate trends for the two sample economies. Results indicate that comparable deficit levels do not guarantee similar debt trajectories, as the interest growth gap and fiscal feedback intensity serve as core constraints for debt sustainability. Both countries meet the mathematical convergence condition for their debt systems, but Japan features a much higher steady state debt ratio that demands sustained fiscal adjustment. A critical upper bound of financing interest rate is identified; exceeding this threshold triggers exponential debt divergence even under fiscal rectification. Persistent primary deficits raise theoretical steady state debt levels and impede the realisation of long-term equilibrium. This study compensates for limitations of static debt models and provides quantitative references for sovereign debt risk evaluation and the optimisation of government financing strategies.

Keywords: government debt sustainability, fiscal feedback mechanism, debt dynamic model, steady-state interest rate

1. Introduction

Japan and Canada are both developed economies. These economies face deepening population ageing and public health crisis impacts, while maintaining long run fiscal deficit positions. However, against this background, there is a significant difference in the degree of growth in the size of their debt. Japan's debt-to-GDP ratio continues to rise, and Canada's debt-to-GDP ratio has always been in the controllable range. It is difficult for static models to describe long-term change trends, and existing models cannot explain the real problem of the different evolution rates of debt-GDP ratios under the same deficit [1]. Therefore, this paper uses a first-order linear differential equation to construct a model to describe the continuous dynamic change model of the debt-GDP ratio and interest rates, and constructs a cross-term optimal financing model.

The research on the sustainability of existing sovereign debt mostly focuses on fiscal deficits, interest rate growth gaps, and fiscal adjustment mechanisms [2]. The government will curb debt expansion by adjusting the basic fiscal surplus [3]. There is financial fatigue in developed

economies, and the financial adjustment efficiency of the financial government will gradually weaken with the accumulation of debt, resulting in the differentiation of debt trends. Extant literature has extensively analysed economy level debt sustainability from the perspectives of fiscal deficits, interest rate growth gaps, and fiscal adjustment mechanisms [4]. There are still research gaps in explaining the evolution and differentiation of debt in developed economies and depicting the continuous dynamic changes in debt ratios against the background of homogeneous fiscal deficits. This paper takes Japan and Canada, which have been in fiscal deficits for a long time but have significant differences in debt growth trends, as a case study. It uses the first-order linear differential equation to establish a cross-period optimal financing model, analyses the dynamic linkage relationship between interest rates, economic growth, and debt ratios between the two countries, and explores the core driving mechanism of the differentiation of debt paths between the two countries. It has theoretical and practical value. In a theoretical sense, this paper adopts the continuous dynamic modelling method, which makes up for the limitations of the traditional static model. In a practical sense, by comparing the debt governance effect of the two countries, it provides reference for countries to optimise financial financing strategies and balance fiscal expansion and debt risk.

2. Basic theory

The theory of dynamic evolution of debt is an important basis for the analysis of this article. For the first-order linear non-uniform differential equation system that depicts the change of the debt-GDP ratio, the integral factor method is a classic analytical means to solve this kind of continuous-time model. Different from the traditional discrete debt accounting idea, the continuous-time differential equation can capture the smooth change of the debt ratio at any time node [5]. In the actual modelling operation, it is necessary to combine the initial conditions given by the initial state of the debt of the economy, substitute the differential equation solving process to determine the integral constant, and derive the general solution of the evolution of the debt ratio over time. On this basis, according to the core connotation of steady-state equilibrium in economics, the steady-state solution of the dynamic system can be solved by setting the derivative of the debt-GDP ratio to zero [6]. The steady-state solution represents the long-term equilibrium level of the economy's debt to the proportion of GDP, which can be directly used to determine whether the entire debt system tends to converge or diverge. At the same time, it is also the core basis for judging whether a country's sovereign debt has long-term sustainability, and provides a theoretical standard for distinguishing the long-term operation trajectory of different countries' debt [7].

The theory of dynamic financing focuses on the economic effect of the continuous adjustment of fiscal policy over time, and takes into account the change of interest rate growth gap over time and the institutional inertia of the government's fiscal adjustment behaviour [8]. Nevertheless, conventional debt sustainability research often treats interest and economic growth rates as exogenous fixed constants. Such work neglects their macro driven dynamic fluctuations and institutional lags constraining fiscal adjustment. From the micro-level of economic decision-making, the financial choice of a country's government is not entirely based on pure economic goals, but is also constrained by many factors such as the domestic political system, voters' demands for people's livelihood, the rigidity of social welfare expenditure, and policy preferences [9]. The above-mentioned differences in the national-level system and social environment will directly change the intensity and adjustment speed of the government's financial feedback, resulting in obvious differentiation of the debt accumulation rhythm of different economies even in the face of similar

fiscal deficits, which is also an important source of the significant difference in the debt development paths of Japan and Canada [10].

3. Model construction

This paper takes time t as an independent variable, and takes the debt-GDP ratio as the core to build a first-order linear differential equation to describe the dynamic evolution of government debt. The core variables in the model include real debt financing interest rate r , real economic growth rate g , and the proportion of basic fiscal surplus to GDPs. On this basis, the Bohn fiscal feedback rule is introduced. Setting s as a function of b to incorporate government behavioural responses yields the debt dynamic equation with fiscal response functions. This paper uses the integral factor method to solve the problem, as shown in Equation (1).

$$\frac{db}{dt} = (r - g - \alpha 1)b + \alpha 0 \quad (1)$$

First, the equation is organised into the standard form of first-order linear differentiation, and the corresponding integral factor is constructed. After multiplying the integral factor on both sides of the equation at the same time, the left side is transformed into a full differential form, as shown in Equation (2).

$$b(t) = b(0)e^{(r-g-\alpha 1)t} - \frac{\alpha 0}{r-g-\alpha 1} \quad (2)$$

Carry out integral operations on both sides of the equation, bring in the initial conditions of the debt ratio at the initial moment, determine the integral constant, and finally obtain the general solution of the debt ratio.

4. Dynamic analysis

In order to carry out a quantitative analysis of the debt dynamics of the two governments, this article selects the World Economic Outlook database released by the International Monetary Fund as the data source. The database has been unified and standardised, has transnational comparability, and is an authoritative data source widely used in empirical research related to sovereign debt. This article extracts the annual macro-time series data of Canada and Japan in the past five years. The variables mainly include the size of government debt stock, debt-GDP ratio, real GDP growth rate, basic fiscal surplus, and debt interest expenditure. In the process of sample screening, this paper gives priority to the actual variables that have been eliminated from inflation to avoid the interference caused by price fluctuations in the estimation of model parameters, and to ensure that the data is consistent with the calibre of theoretical variables such as real interest rates and actual economic growth in the differential model. Outliers characterised by abnormal fluctuations are removed to guarantee time series data stability. According to this standardised macro data set, the measurement and estimation of the financial feedback coefficient can be completed, which can provide reliable initial parameters for subsequent debt path simulation, so that the simulation results of the differential equation model can fit the real macroeconomic operation background of the two countries. Based on the setting of Bohn's fiscal feedback rules, with the proportion of basic fiscal surplus to GDP as the explained variable and the debt-GDP ratio as the explanatory variable, simple linear regression is carried out for the two countries respectively. The fiscal feedback coefficient and basic policy constant are

estimated. As seen in Table 1, based on the data of the sample starting year, the initial debt ratio b_0 , the average real GDP growth rate g , and the average real debt interest rate r are determined.

Table 1. The parameters of Japan and Canada

Parameters	Value (Japan)	Value (Canada)
r	0.005	0.022
g	0.010	0.020
α_1	0.04	0.055
α_0	-0.062	-0.018
b_0	2.62	0.71

Substitute the steady-state condition into the debt dynamic equation after the introduction of the fiscal response function, and combine it with the system convergence constraint $r-g-\alpha_1 < 0$ to solve the steady-state optimal loan interest rate that maintains the long-term sustainability of debt. The interest rate represents the upper limit of the convergent interest rate that the debt system can maintain. When the actual debt financing interest rate of the economy is higher than the critical value, the debt ratio will show the characteristic of exponential divergence, and the debt risk will continue to increase; if the actual interest rate is lower than the critical level, the debt ratio will gradually converge to the steady-state equilibrium position, and the debt will have a sustainable continuation. It is calculated that Japan's $r^*=0.05$, Canada: $r^*=0.075$. In reality, the actual interest rates of both countries are far lower than the upper limit, which meets the mathematical convergence conditions of the model. Japan's steady-state debt ratio is positive, which has a theoretical economic meaning, but the steady-state value is relatively high, which means that high-term financial feedback needs to be maintained to achieve this balance. Substitute the two sets of national parameters into the debt ratio formula to carry out long-term trajectory simulation of the debt of the two countries. Visually compare the differences in the debt evolution paths of the two economies. From the simulation trajectory in Figure 1, it can be observed that although there is a fiscal feedback mechanism in both countries, the evolution of long-term debt is obviously differentiated due to the difference in interest rate-growth gap and financial feedback intensity, which also explains the difference in the debt performance of the two countries in reality.

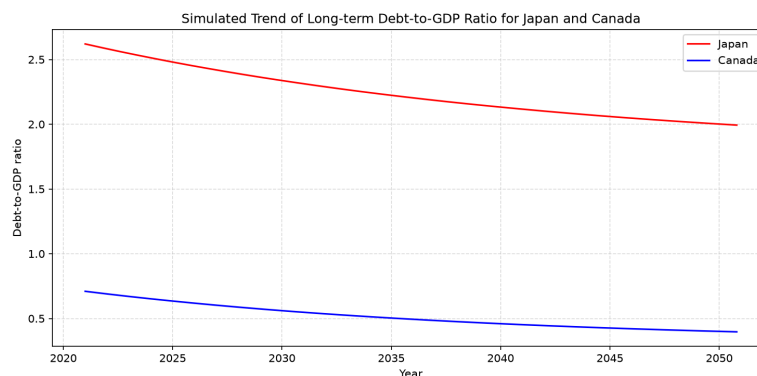


Figure 1. Japan and Canada's long-term debt-to-GDP ratio (picture credit: original)

5. Conclusion

This paper constructs a dynamic differential equation of government debt with a fiscal feedback mechanism, and conducts a quantitative analysis of debt sustainability with Canada and Japan as research samples. Research shows that the relative difference between interest rates and real economic growth alone cannot explain the economic fact that there is a significant differentiation in the long-term evolution of debt in Japan and Canada. The interest rate-growth gap will have an important impact on the cumulative rate of debt, but the implementation of the government's financial feedback, that is, the willingness and ability of the government to take the initiative to adjust the basic fiscal surplus when the debt burden rises, is also a key factor in determining the long-term trend of debt. By analysing and solving differential equations, this paper derives the upper limit of the critical financing interest rate to maintain the long-term sustainable operation of a country's debt, which is determined by the economic growth rate and the financial feedback coefficient. Model results indicate that once an economy's actual debt-financing interest rate exceeds this critical threshold, the debt-to-GDP ratio exhibits exponential divergence despite formal fiscal-adjustment frameworks and consolidation policies. This pattern drives rising debt burdens and growing sovereign-debt risks, and the risk of sovereign debt will continue to increase. Simulation results show that the two sample countries, Japan and Canada, meet the mathematical convergence conditions of the debt system at the parameter level, but the persistent basic fiscal deficit will push up the theoretical steady-state debt level, making it difficult to achieve the theoretical equilibrium state in the real economic environment, which is an important reason why Japan is in the deep debt dilemma.

The main contribution of this article is to combine the benchmark debt evolution model with the financial feedback rule, obtain the critical interest rate conditions of the debt general solution, steady-state equilibrium, and debt sustainability through analysis, complete the comparative simulation analysis of the two economies, and make up for the lack of dynamic mechanism in debt ratio analysis. The modelling ideas of this article can provide an analytical framework for the analysis of similar transnational debt problems, which is convenient for other researchers to carry out parameter adjustments. This article also has certain limitations. For instance, fiscal feedback is specified with a fixed-linear functional form. The model omits frequently-occurring exogenous stochastic economic shocks observed in real-world contexts, and does not add the inherent adjustment stickiness of the fiscal policy itself, as well as the time-changing characteristics of the parameters brought about by institutional changes and social structure changes over time. In view of the above shortcomings, future relevant research can expand the model into a second-order sticky adjustment model, introduce random disturbance terms in the equation, restore the lag effect prevalent in the process of adjusting real fiscal policy, improve the ability of the model to fit the real debt evolution path of countries, and enhance the effectiveness of the model's interpretation of real problems.

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