

A Comparative Study of Partial Differential Based Equation and Stochastic Approaches to Option Pricing

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Abstract. This study compares two numerical approaches for European option pricing: a partial differential equation (PDE)-based approach and a stochastic approach. The PDE-based approach is implemented through the Crank–Nicolson finite difference method. Using a selected Apple Inc. (AAPL) European call option and consistent market parameters, the stochastic approach estimates the option value through Monte Carlo simulation. The analytical Black–Scholes solution is used as a theoretical benchmark to evaluate the pricing performance of both methods. The numerical results show that both approaches produce option prices that are highly consistent with the analytical benchmark. The PDE-based method generates a stable and deterministic numerical solution. And the stochastic approach achieves a highly accurate estimate. In the selected experiment, although this difference does not imply that the stochastic approach is universally more accurate, the stochastic estimate is slightly closer to the analytical benchmark. Overall, the comparison demonstrates that both approaches are effective for European option pricing, with the PDE-based approach offering greater numerical consistency and the stochastic approach providing greater flexibility for more complex pricing applications.

Keywords: Black–Scholes model, Option Pricing, Partial differential equation

1. Introduction

Option pricing is one of the most important parts in mathematical finance. There are also many mathematics and statistics ways to analyze these significant properties. These main characteristics provide theoretical foundations for valuing financial derivatives and managing financial risk. Accurate option pricing is essential for investors, financial institutions, and market regulators, because it supports investment decisions, risk management, and derivative trading. Since the introduction of the Black–Scholes model in 1973 [1], mathematical methods have played a central role in option pricing. As financial markets have become increasingly complex, there are lots of different approaches to improve pricing accuracy and computational efficiency.

Among the different kinds of methods, Partial Differential Equation (PDE)-based approaches and stochastic approaches are two of the most widely adopted frameworks [2]. PDE-based methods formulate option pricing as a deterministic mathematical problem. The stochastic approaches model focuses a random evolution of asset prices by using stochastic processes and simulation techniques. Both approaches have been extensively applied in academic research and financial practice. For

example, PDE-based methods are commonly applied to standard European option pricing [3], and stochastic approaches are widely used for pricing complex derivatives.

Previous studies have established one of the fundamental approaches to option pricing by using PDE-based methods. Black and Scholes proposed the classical Black–Scholes model [4]. It formulates the option pricing as a partial differential equation and provides the theoretical foundation for option pricing [5]. Depending on this framework, researchers have developed lots of different numerical methods like finite difference methods. Those frameworks make it be easier to figure out the Black–Scholes equation in a more efficiently way during practical applications.

To address more complex pricing problems, Boyle introduced Monte Carlo simulation into option pricing [6]. It allows stochastic asset price models to be evaluated through random path simulation. Subsequent studies extended Monte Carlo methods to the pricing of high-dimensional and path-dependent derivatives. Those demonstrate a greater modelling flexibility than traditional PDE-based approaches. These studies also reported that Monte Carlo methods generally require substantially more computational time to achieve comparable pricing accuracy [6]. Overall, existing research suggests that PDE-based approaches are computationally efficient to deal with relatively simple option pricing problems. And the stochastic approaches are more flexible on solving modelling realistic market dynamics and complex financial derivatives.

Although both PDE-based and stochastic approaches have been widely studied, the existing research has primarily focused on improving individual pricing models or numerical algorithms [7]. Under different modelling assumptions and evaluation criteria, comparisons between these two different approaches are often conducted. Sometimes, it is hard to directly evaluate their relative strengths and limitations, but it could provide a clearer comparison of their mathematical characteristics, computational efficiency, pricing accuracy. The practical applicability can get a more valuable consistent framework.

Thus, providing a comparative analysis of partial differential equation and stochastic approaches to option pricing is important for method data analysis. Specifically, it will examine the theoretical foundations of the two approaches and compare their numerical performance by using a common option pricing problem. This comparison focuses on computational efficiency, pricing accuracy, and practical applicability. By identifying the advantages and limitations of each approach, the result seeks to provide a clearer understanding of their respective roles in option pricing and offer useful insights into selecting appropriate mathematical methods for different financial applications.

2. Method

2.1. Dataset preparation

The comparison between the partial differential equation (PDE)-based approach and the stochastic approach evaluated by using the same option pricing problem and identical model parameters. To get a consistent comparison between these two approaches, both methods are calculated with the same option contract and market conditions. Apple Inc. (AAPL) is selected as the underlying asset because its stock and options are actively traded, and sufficient market information is publicly available [8]. The market observation date used in this study is July 31, 2026.

Table 1. Market data and model inputs for the selected AAPL option

Variable	Symbol	Value	Description
Observation date	/	July 31, 2026	Market data date

Table 1. (continued)

AAPL stock price	S_0	\$308.91	Closing stock price
Strike price	K	\$310.00	Selected call strike
Expiration date	/	September 18, 2026	Option expiration
Time to maturity	T	49 days/0.1342 years	Calculated from dates
Volatility	σ	/	Implied volatility
Risk-free rate	r	3.77%	Short-term Treasury Proxy

Table 1 summarizes the market data and model parameters used in the numerical experiment. The selected AAPL European call option has a strike price of \$310, which is close to the observed stock price of \$308.91. It makes a contract approximate at-the-money. With an expiration date of September 18, 2026, the option has 49 days remaining to maturity. It corresponds to approximately 0.1342 years. In addition to the first column variables, the option pricing models require to estimate the volatility. These data imply that the volatility of the selected AAPL option is used as the volatility input in a short-term U.S. Treasury rate is used as a proxy for the risk-free interest rate.

To ensure a consistent comparison, the same parameter values are used for both the PDE-based and stochastic approaches. Holding the market inputs as a constant, the two methods on the same European call option price under identical conditions. Therefore, the differences in option prices can be attributed primarily to the numerical characteristics.

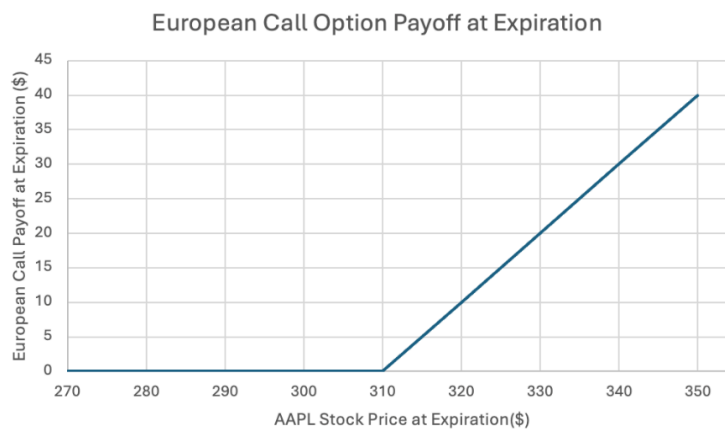


Figure 1. Selected AAPL call option's payoff structure (picture credit: original)

Figure 1 illustrates the payoff structure of the selected AAPL European call option. It underlies the stock price is approximately \$308.91, and a strike price is \$310. The sharp increase showed in both vertical and horizontal axis also compares the PDE-based benchmarks and stochastic pricing approaches. The selected market parameters provide a common experimental setting for the comparison. As the theoretical benchmark, the analytical Black–Scholes solution trend is subsequently increasing. For the PDE-based and stochastic approaches, those values and trends independently estimate the accuracy of the same European call option. This graph allows the pricing performance and numerical characteristics to be compared under identical input condition.

2.2. Partial differential equation based approach

The PDE-based approach adopted in this study is based on the classical Black–Scholes option pricing model. It formulates the option pricing problem as a second-order partial differential equation. Assuming the underlying asset follows a geometric Brownian motion under a risk-neutral measure and that the market is frictionless [9], the Black–Scholes framework provides a deterministic mathematical model to analysis the European option pricing.

The governing Black–Scholes equation is expressed as [10],

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (1)$$

Where $V = V(S, t)$ represents the value of the option with function of asset price and time, S denotes the price of the underlying asset, t represents time, r is the continuously compounded risk-free interest rate, and σ represents the volatility of the underlying asset. The derivatives $\frac{\partial V}{\partial t}$ and $\frac{\partial V}{\partial S}$ describe the sensitivity of the option value to changes in time and the underlying asset price, respectively, while $\frac{\partial^2 V}{\partial S^2}$ captures the curvature of the option value with respect to the underlying asset price.

Since a closed-form analytical solution may not be available for more general option pricing problems, this study applies the finite difference method (FDM) to solve the Black–Scholes partial differential equation numerically [11]. The method discretizes the underlying asset price S and time t into finite grids. The derivatives in the Black–Scholes equation, including the time derivative and the first and second order derivatives to the asset price, are then approximated at each grid point. In this study, the Crank–Nicolson finite difference scheme is used to transform the continuous Black–Scholes equation into a system of algebraic equations [12], from which the option value can be obtained numerically.

The PDE-based approach is implemented using the AAPL option parameters described in Section 2.1, including the underlying stock and strike price, risk-free interest rate, volatility, and time to maturity. The resulting numerical option price is used as one of the main results in the comparative analysis. It is first compared with the analytical Black–Scholes benchmark to evaluate pricing accuracy and is then compared with the stochastic approach introduced in Section 2.3. This allows the two approaches to be evaluated under the same market conditions in terms of pricing accuracy and numerical characteristics.

2.3. Stochastic approach

The stochastic approach adopted in this study is based on Monte Carlo simulation. It estimates option prices by simulating the random evolution of the underlying asset price. Unlike the PDE-based approach, which solves a deterministic mathematical equation, the stochastic approach models the uncertainty of financial markets directly through stochastic processes. This method is particularly suitable for pricing derivatives with complex payoff structures or path-dependent characteristics, where analytical or PDE-based solutions may be difficult to obtain.

Under the risk-neutral measure, the underlying asset price is assumed to follow a geometric Brownian motion [13]. This asset price dynamics are described by the stochastic differential equation,

$$dS_t = rS_t dt + \sigma S_t dW_t \quad (2)$$

where S_t represents the underlying asset price at time t , r is the risk-free interest rate, σ represents the volatility of the underlying asset, and W_t denotes a standard Brownian motion that introduces randomness into the asset-price process. The first term $rS_t dt$, represents the expected growth of the asset price under the risk-neutral measure. And the second term $\sigma S_t dW_t$ captures the random fluctuations in the asset price caused by market uncertainty.

Depending on the formula, the stochastic approach is implemented using the same AAPL option parameters described in Section 2.1 to ensure consistency with the PDE-based approach. Based on the stochastic approach, many possible asset-price outcomes are simulated. For the different simulated outcomes, corresponding option payoff is calculated and discounted to its present value at maturity. The stochastic option price is estimated by averaging discounted payoffs for simulated outcomes [14].

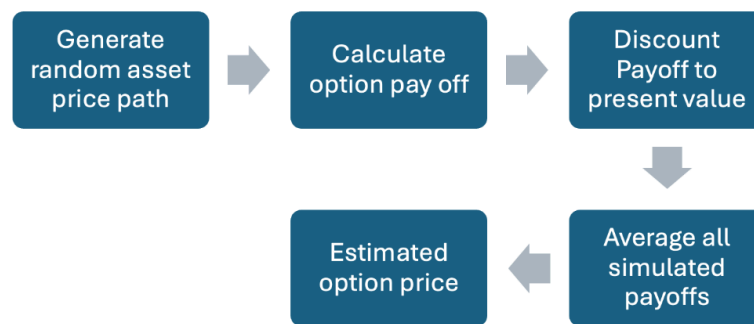


Figure 2. Procedure of the stochastic approach for option pricing (picture credit: original)

Figure 2 summarizes the computational procedure of the stochastic approach. The first step is generating random asset-price outcomes to determine the corresponding option payoffs. Then, applying discount payoff to present value. After that, the process focuses on getting all average simulated payoff to the estimated option price. Then, comparing PDE-based approach can get the results from Monte Carlo simulation. This approach can assess the strengths and limitations of each method under identical experimental conditions. The resulting stochastic option price can be compared with both analytical Black–Scholes benchmark and numerical result obtained from the PDE-based approach [15]. This comparison allows the two numerical approaches to be examined under the same AAPL market conditions.

2.4. Evaluation metrics

To compare the PDE-based approach and the stochastic approach, there are three important evaluation criteria which are pricing accuracy, numerical methods analysis and practical applicability. Depending on these criteria, the data analysis could provide a more systematic framework for assessing the performance of both methods under the same market conditions.

Firstly, pricing accuracy is evaluated by comparing the option prices. This analysis is generated by the two numerical methods with the analytical Black–Scholes solution. So, it is important to get the different benchmarks for the selected European call option. Then calculating the absolute pricing error to get the difference between numerical result and analytical price, where smaller error indicates a closer approximation to the theoretical solution.

Second, the numerical characteristics of the two approaches are compared. This comparison considers the fundamental properties of each method including whether the solution is deterministic

or simulation based. It also includes the primary source of numerical error and the consistency of repeated calculations under identical parameter settings. Finally, the practical applicability of two numerical methods is discussed based on the experimental findings and the characteristics. The PDE-based approach provides a structured and deterministic framework for solving the Black–Scholes pricing equation. In simulating the difference between asset-price dynamics and payoff structures, stochastic approach provides a greater flexibility. This comparison helps identify the strengths and limitations of each method in practical option pricing applications. The evaluation results are presented and discussed in Section 3. Both approaches are compared using the same AAPL option data and market parameters.

3. Results and discussion

3.1. Pricing accuracy

The pricing accuracy of the partial differential equation-based approach and the stochastic approach was evaluated using an Apple Inc. European call option. The strike price is \$310 with an expiration date of September 18, 2026. The underlying AAPL stock price was \$308.91. It implied volatility and risk-free interest rate which were set at 28.06% and 3.77% respectively. The remaining time to maturity was approximately 49 days.

Then, for the analysis of the analytical Black–Scholes solution, it was used as the theoretical benchmark. Based on the selected parameters, the analytical value was calculated as \$12.8937. PDE-based price was obtained by 800 asset-price intervals and 800-time steps by using Crank–Nicolson finite difference method. Targeted stochastic price was estimated using Monte Carlo simulation with 2,000,000 simulated asset prices during a fixed random seed of 2026. This simulation ensures the reproducibility. For the overall analysis, the unrounded Monte Carlo absolute error is 0.0000087 and the relative error is 0.000067%.

Table 2. Pricing accuracy of the option pricing methods

Method	Option Price (\$)	Absolute Error (\$)*	Relative Error(%)*
Black–Scholes Benchmark	12.8937	0.0000	0.0000
Crank–Nicolson Finite Difference Method	12.8931	0.0006	0.0047
Monte Carlo Stochastic Simulation	12.8937	0.0000*	0.0001*

As shown in Table 2, both numerical methods produced their own option values. These methods were extremely close to the analytical Black–Scholes benchmark. The finite difference method generated a price of \$12.8931. It corresponds to an absolute pricing error of approximately \$0.0006 and a relative error of 0.0047%. And the stochastic approach produced estimated price of \$12.8937 with an absolute error. The accuracy of the error is less than \$0.0001 in this simulation. The 95% confidence interval was approximately \$12.8662 to \$12.9213 which contained the analytical Black–Scholes value.

Table 3. Uncertainty of stochastic price estimate

Measure	Value
Number of simulations	2,000,000
Standard Error	0.0140
95% confidence level	[12.8662, 12.9213]
Random Seed	2026

Table 3 summarizes the uncertainty associated with the stochastic price estimate. With 2,000,000 simulations, the estimated standard error is 0.0140, producing a 95% confidence interval of [12.8662, 12.9213]. The analytical Black–Scholes value of \$12.8937 falls within this interval. It indicates that the stochastic estimate is consistent with the theoretical benchmark.

These results show that the PDE-based approach provides a deterministic solution for a fixed numerical grid. And the stochastic approach remains subject to sampling variation, because it is implemented through Monte Carlo simulation [16]. Therefore, the particularly small pricing error observed for the stochastic approach in this experiment should not be interpreted as evidence. It is more accurate than the PDE-based approach. Instead, the estimated option price may vary slightly when a different random seed or a different number of simulation paths is used.

3.2. Comparative analysis of the two approaches

The results in Section 3.1 show that both the partial differential equation-based approach and stochastic approach produced option prices. They were very close to the analytical Black–Scholes benchmark. This benchmark price was calculated as \$12.8937, and the finite difference method generated an option price of \$12.8931. It corresponds to an absolute pricing error of approximately \$0.0006. The stochastic approach simulation's estimated option price is approximately \$12.8937 with an absolute error. This error is less than \$0.0001 in the selected simulation.

The stochastic approach estimate sometimes was closer to the analytical benchmark in the experiment. But when focusing on the finite difference method, it produces a more accurate deterministic numerical solution. Once the asset-price grid, time grid, boundary conditions, and model parameters are fixed, repeated calculations generate the same option price. Its pricing error mainly results from the discretization on continuous Black–Scholes partial differential equation.

In contrast, stochastic approach estimates option price through random simulation. In this study, Monte Carlo simulation generates many possible asset-price outcomes, and option value is estimated from the average discounted payoff. As a result, repeated simulations may produce slightly different option prices even when the market parameters remain unchanged. The particularly small pricing error observed in this experiment may therefore partly reflect the selected random seed and a lot of simulations.

Thus, the results reflect that both approaches provide highly accurate estimates for the selected AAPL European call option. The PDE-based approach offers a stable and reproducible numerical result. And the stochastic approach provides an accurate statistical estimate that remains subject to sampling variation. So, the primary difference between the two approaches is the numerical mechanisms, which employ to obtain the final option value in their ability to price European options accurately.

Table 4. Summary comparison of the two approaches

Feature	PDE-Based Approach	Stochastic Approach
Method	Finite difference method (Crank–Nicolson finite difference method)	Monte Carlo simulation
Calculated option price	\$12.8931	Approximately \$12.8937
Analytical benchmark	\$12.8937	\$12.8937
Main source of error	Grid discretization	Random sampling variation
Nature of result	Deterministic	Simulation-based
Repeated result	Identical under fixed settings	May vary slightly between runs

Table 4 summarizes the main numerical differences between the two approaches. The PDE-based approach obtains the option price by numerically solving the governing pricing equation. And the stochastic approach estimates the option value through repeated random simulations. Therefore, both approaches achieve a similar pricing accuracy in this experiment. But under the nature and repeated result, they differ substantially in the way numerical approximation and uncertainty are handled.

3.3. Overall discussion

Overall, the results demonstrate that both the PDE-based approach and the stochastic approach can get accurate estimates for the selected AAPL European call option. With the data parameter settings, the stochastic approach produced an estimate which is slightly closer to the analytical Black–Scholes benchmark. Even the difference in pricing accuracy was very small, both approaches can be considered effective for the pricing problem examined in this study.

What is more, it is also important to compare the trade-off between numerical stability and simulation flexibility. The PDE-based approach provides a deterministic and reproducible result when the numerical grid and model parameters are fixed. In contrast, the stochastic approach produces a statistical estimate and is therefore affected by sampling variation. The number's simulations increase can improve the stability of stochastic estimate, even the result remains dependent on the simulated sample.

From a broader perspective, the stochastic approach may be considered more flexible because it can be extended to derivatives with path-dependent payoffs, multiple underlying assets, or more complex sources of uncertainty. The PDE-based approach is more direct and stable for standard low-dimensional options with clearly defined boundary conditions. Therefore, although the stochastic approach performed slightly better in terms of pricing accuracy in this experiment, the most appropriate method depends on the structure and complexity of the option pricing problem.

4. Conclusion

The study compared the PDE-based approach and the stochastic approach for option price analysis. The PDE-based approach applied the Black–Scholes Benchmark, Crank–Nicolson finite difference and Monte Carlo Stochastic Simulation methods to solve the option pricing accuracy. With the Monte Carlo simulation, stochastic approach estimates the discounted expected payoff more

accurately. Both methods were evaluated using the same selected AAPL option and market parameters. The analytical Black–Scholes solution used as the theoretical benchmark.

Thus, the results show that both approaches provide highly accurate option price estimates. The PDE-based approach produced a price of \$12.8931 compared with the analytical benchmark of \$12.8937. And for the stochastic approach, it produced an estimate of approximately \$12.8937 in the selected simulation. Although the stochastic approach was slightly closer to the benchmark in this experiment, the result remains to random sampling variation. In contrast, the PDE-based approach provides a deterministic and reproducible result under fixed numerical settings. Overall, both approaches demonstrated strong pricing performance for the selected AAPL European call option. In this experiment, the stochastic approach achieved slightly higher pricing accuracy and showed greater flexibility. Relatively, the PDE-based approach provided greater numerical consistency and reproducibility. Therefore, regardless the preferred method may vary depending on the characteristics of the pricing problem, the stochastic approach showed a advantage in the selected experiment.

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