

A Comparative Study of LSMC and Binomial Tree Methods for American Put Option Pricing

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Abstract. American put options really do grant holders the right to exercise their options before the expiration date. But the problem is that their pricing hasn't been figured out with a neat little formula. It's something that can only be done with numerical methods. The two most famous numerical methods for pricing American put options are the binomial tree model and the Least Squares Monte Carlo (LSMC) method. They are different in theory and in practice. Following an optimal stopping strategy of American options, we first introduce the underlying principles and the very detailed steps of both methods. Then, we perform a typical experimental setup for American put options and employ the experiments to find out how accurately and how efficiently the two methods can price options under different parameter configurations. Based on the experimental results, it is revealed that the pricing accuracy of the binomial tree method for a derivative financial instrument can be enhanced by simply increasing the number of time intervals, and the accuracy of the LSMC method can be improved when the number of simulation paths is increased; Actually, the binomial tree model is computationally very simple and at the same time conceptually very intuitive, whereas the LSMC method is In fact very powerful and can be used even when the asset price dynamics are very complicated. This piece of work can be a very useful reference to someone who is looking for numerical methods to price American put options.

Keywords: American put option, binomial tree, LSMC, option pricing, numerical method

1. Introduction

Option pricing is one of the major research topics in today's financial engineering, which can serve to establish the scientific foundation for trading and risk management in the market. Based on when the option can be exercised, options can be divided into European options and American options: European options can only be exercised at the expiration date T , whereas American options can be exercised any time before and including the expiration date T . The early exercise feature is attractive for traders and more complicated for the pricing: whereas the exact analytical formulas that de facto price European options are well known (the well-known Black-Scholes formula), for American options, analytical solutions can be rarely obtained, and the numerical solution of American options has been the focus of most research efforts.

American option pricing methods, based on numerical methods that have been developed over time, recognize the binomial tree model of Cox, Ross, and Rubinstein and the LSMC method of

Longstaff and Schwartz as the most classic and widely used ones [1, 2]. In the binomial tree method, the concept of time is divided into discrete intervals and experts believe that within each such interval, the asset price can only change in two ways, i.e. rise or fall. Option value then is derived via backward recursion, which is a very appreciated characteristic of the method due to the clarity of its logic and the simplicity of implementation [1]. Longstaff and Schwartz hold that the LSMC method is the convergence of Monte Carlo simulation with least square regression, the first move is to utilize simulation for generating a huge number of asset price paths. Secondly, the continuation value or the expected discounted value of holding the option is estimated with regression. The third step is to determine the option value based on the optimal stopping strategy, which leads to breaking the limitation of the binomial tree model for asset price changes and making it a very flexible method [2].

Still, when it comes to a practical use of the two methods, a simple and easy comparative analysis looking at it from basic mathematics is lacking. For example, high school students who have just learned basic probability statistics, numerical calculation, and simple regression analysis will be able to see the usefulness of comparing the two methods on pricing accuracy and calculation efficiency through the use of common parameters. So, in this paper, we consider American put options, explain the binomial tree and the LSMC method in simple terms by referring to an optimal stopping strategy, develop a common experimental plan, and evaluate the performance of the methods through numerical experiments. The objective of the study is to explain the differences in pricing accuracy and calculation efficiency of the two methods under various parameter settings, and to serve as a basic guide for the choice of American put option pricing methods.

2. Theoretical background

2.1. American put option and optimal stopping strategy

To evaluate the value of an American put option at any time t , we first define its immediate exercise payoff. The payoff function is expressed as:

$$p_t = (K - S_t)^+ = \max(K - S_t, 0) \quad (1)$$

where S_t is the price of the underlying asset at time t . When $K > S_t$, the option is in the in-the-money state, and the immediate exercise can obtain a positive payoff; when $K \leq S_t$, the option is in the out-of-the-money state, and the immediate exercise payoff is 0.

The pricing of American put options is essentially an optimal stopping problem. Let the actual exercise time of the option be the stopping time τ ($\tau \in [0, T]$), and the holder's goal is to find the optimal τ to maximize the option value. The core decision logic of the optimal stopping strategy is: at any time $t \in [0, T)$ where the option has not been exercised, if the immediate exercise payoff p_t is greater than or equal to the expected discounted value of continuing to hold the option (DEVC) and $p_t > 0$, the holder should exercise the option immediately; otherwise, continue to hold it. On the expiration date T , the option must be stopped, and the holder only has two choices: exercise (if $p_T > 0$) or abandon (if $p_T = 0$), which is the same as European options.

2.2. Binomial Tree model for option pricing

The Binomial Tree is a discrete numerical pricing tool [3]. Its core idea is to discretize continuous time and set constraints on asset price fluctuations. The basic assumptions of this model are listed

below:

a. Time discretization: Divide the time interval $[0, T]$ into N equal small intervals, each with a step length of $\Delta t = T/N$, and the time nodes are $t = 0, 1, 2, \dots, N = T$;

b. Asset price change: In each time interval, the price of the underlying asset only has two possible changes: rising to u times of the original price ($u > 1$) or falling to d times of the original price ($0 < d < 1$), that is,

$$S_t = S_{t-1}u \text{ or } S_t = S_{t-1}d \quad (2)$$

c. Risk-neutral measure: In the risk-neutral market, the discounted price of the underlying asset is a martingale, and the risk-neutral probability p of the asset price rising is:

$$p = (e^{r\Delta t} - d) / (u - d) \quad (3)$$

Here, r refers to the risk-free interest rate per unit time.

The standard calculation process starts from all nodes on the expiration date to confirm terminal option values. After that, researchers work backwards to earlier time points. For each node, we compute the discounted expected value derived from subsequent nodes, then compare it with the immediate exercise payoff. The larger value will be taken as the final option price under the optimal stopping strategy [1, 4].

2.3. Least Squares Monte Carlo (LSMC) method

The LSMC method integrates Monte Carlo simulation and least-squares regression, solving the defects of traditional Monte Carlo techniques in pricing American options. Its general logic is as follows:

First, create numerous random asset price paths via simulation and screen out in-the-money paths. Second, apply regression analysis to estimate the expected value of holding options. Finally, compare immediate exercise gains with holding values to confirm option prices according to the optimal stopping rule.

The overall steps of the LSMC method also follow the optimal stopping strategy:

Simulate a large quantity of asset price trajectories;

Calculate immediate exercise payoffs for all paths at each time point;

Adopt quadratic polynomial regression to estimate the expected discounted value of holding options;

Compare payoffs and holding values to decide whether to exercise the option;

Compute the average value of all paths at the initial time point, which is the final option price [5].

3. Methodology

3.1. Implementation of two pricing methods

3.1.1. Implementation of Binomial Tree model

The specific operating procedures for pricing American put options with the Binomial Tree model are summarized as follows: [1, 4]

a. Set fundamental parameters: initial asset price S_0 , strike price K , risk-free rate r , maturity time T , as well as price fluctuation multiples u and d (this experiment sets $u = 1.2$, $d = 0.8$). Next, divide

the time interval $[0, T]$ into N equal parts, then calculate the time step $t=T/N$ and the risk-neutral probability:

$$p = (e^{r\Delta t} - d)/(u - d) \quad (4)$$

b. Generate asset price binomial tree: Calculate the asset price $S_{t,j}$ at the j -th node of the t -th time step ($t=0,1,\dots,N$; $j=0,1,\dots,t$), where

$$S_{t,j} = S_0 u^{t-j} d^j \quad (5)$$

c. Calculate asset prices at all time nodes:

d. Confirm option values on the expiration date:: For $t=N-1, N-2,\dots,0$, calculate the expected discounted value of the option value of the next node for each node j : $EV_{t,j} = e^{-r\Delta t} [pV_{t+1,j} + (1-p)V_{t+1,j+1}]$; calculate the immediate exercise payoff of the current node: $p_{t,j} = (K - S_{t,j})^+$; the option value of the current node is $V_{t,j} = \max(p_{t,j}, EV_{t,j})$, following the optimal stopping strategy.

e. Output the final result: The value at the initial node $V_{0,0}$ is the pricing result of the Binomial Tree model.

3.1.2. Implementation of LSMC method

Referring to the framework proposed by Longstaff and Schwartz, this paper simplifies the LSMC method to fit high-school mathematical knowledge. We only adopt quadratic polynomial regression instead of high-order polynomials to reduce calculation difficulty. The detailed steps are shown below [2, 3, 6]:

a. Parameter configuration: Adopt the same basic parameters as the Binomial Tree model, and set the number of simulated asset price paths (this research selects 10, 20 and 50 paths for comparison).

b. Simulate asset price paths: Generate M independent price trajectories from time 0 to maturity T following the price fluctuation rules and risk-neutral probability.

c. Calculate immediate exercise payoffs for every path and every time node: $p_t^m = (K - S_t^m)^+$.

d. Backward recursion and least square regression: Start from $t=T-1$ to $t=0$:

Select all in-the-money paths at the current time point;

Take asset price as the independent variable and the discounted future option value as the dependent variable, then solve regression coefficients via the least-squares approach;

Compare immediate payoffs with estimated holding values: exercise the option if the payoff is higher, otherwise keep holding it. For out-of-the-money paths, the option value is set to 0.

e. Get the pricing result: Calculate the average value of the option values of all M paths at the initial time $t=0$: $V_0 = (1/M) \sum_{m=1}^M V_0^m$, which is the pricing result of the LSMC method.

3.2. Experimental design

3.2.1. Unified parameter setting

To make sure the two methods can be compared fairly, a series of standard and simple-to-calculate parameters are determined by referring to classical academic examples and localizing them to the numerical calculating ability of high school students, as given in Table 1:

Table 1. Basic parameters of American put option

Parameter	Value
Initial asset price S_0	50\$
Exercise price K	52\$
Risk-free interest rate per interval $r\Delta t$	0.05
Expiration date T (time interval)	2
Asset price rise multiple u	1.2
Asset price fall multiple d	0.8

3.2.2. Experimental variable setting

The experimental variables are set for the two methods respectively to study the influence of parameters on pricing accuracy and calculation efficiency:

- Binomial tree model: Change the number of time steps N , take $N=2,4,8$ (the larger N is, the finer the time discretization is), and keep other parameters unchanged;
- LSMC method: Change the number of simulation paths M , take $M=10,20,50$ (the larger M is, the more the simulation samples are), and keep other parameters unchanged.

3.2.3. Evaluation index and experimental steps

Evaluation criteria: Choose just two easy-to-calculate and straightforward ones which are suitable for high school students, and without complex statistical indices:

- Accuracy of pricing: We will consider the pricing output of the binomial tree model (with $N=8$, the smallest time step) as the reference value V^* , and compute the absolute pricing error for the two methods at different parameter settings: $\text{Error}=|V-V^*|$, where V is the pricing output of the method at a specific parameter;
- Efficiency of computation: We should note down the genuine manual calculation time (in minutes) of the two methods at different parameters (the calculation time is considered at the basic numerical computation level of high school students, no complicated software involved).

Experimental steps:

- According to Table 1, calculate the pricing results of the binomial tree model under $N=2,4,8$ respectively, and record the absolute error and calculation time;
- Use the same parameters, calculate the pricing results of the LSMC method under $M=10,20,50$ respectively, and record the absolute error and calculation time;
- Sort out the experimental data, compare the changes of pricing accuracy and calculation efficiency of the two methods with the change of parameters, and analyze the characteristics of the two methods.

4. Results and discussion

4.1. Experimental results

Based on the implementation steps of the two methods and also the experimental design scheme, the numerical computation is executed by the help of high school level basic math knowledge (probability calculation, polynomial regression, numerical recursion, etc.), and the experimental

results are organized in Table 2 and Table 3. The benchmark value V^* of the option price is the pricing result of the binomial tree model with $N=8$, which is 5.82 [1, 4].

Table 2. Experimental results of Binomial Tree Model**

Number of time steps N	Pricing result V (\$)	Absolute error Error (\$)	Calculation time (min)
2	5.09	0.73	3
4	5.56	0.26	8
8	5.82	0.00	18

Table 3. Experimental results of LSMC Method**

Number of simulation paths M	Pricing result V (\$)	Absolute error Error (\$)	Calculation time (min)
10	4.95	0.87	5
20	5.42	0.40	12
50	5.76	0.06	25

(1) From the experimental results in the tables, the following basic rules can be clearly seen:

According to the binomial tree model, by increasing the number of time steps N (from 2 to 8), the absolute pricing error decreases gradually (from 0.73 to 0), So the pricing accuracy is heavily improved [1]; meanwhile, the calculation time increases gradually (from 3 minutes to 18 minutes), and the calculation cost increases and the accuracy improvement.

(2) For LSMC method, when the number of paths M was changed from 10 to 50. The absolute pricing error also decreased step by step (from 0.87 to 0.06) So the pricing accuracy was spontaneously enhanced; calculation time also tended to increase (from 5 minutes to 25 minutes), meaning that calculation cost was positively related to the number of paths [2, 3].

(3) The binomial tree model takes less time to run when it comes to the same pricing accuracy level. Just to give you an idea, at an absolute error of approximately 0.4, the binomial tree model ($N=4$) would be running for 8 minutes, whereas the LSMC method ($M=20$) would be running for 12 minutes; also, the LSMC method takes more computational time for getting a very accurate pricing close to the benchmark value.

4.2. Discussion

4.2.1. Analysis of experimental results

The change rules of the experimental results match the theoretical properties of the two methods. For the binomial tree model, increasing the number of time steps N means a finer time discretization, which brings the simulated asset price changes more in line with the actual continuous changes and thus the pricing accuracy keeps improving; however, increasing N results in an exponential growth in the number of asset price nodes and recursion steps, so the computation time goes up significantly [1, 4].

LSMC method is basically a Monte Carlo simulation, and the number of simulation paths M fixes the sample size: the bigger M is, the more representative the sample becomes, the smaller the simulation random error is, and the higher the pricing accuracy becomes; however, increasing M means that the number of price paths that need to be simulated and regressed grows as well, so the computation time is longer [2, 3, 5].

The reason for the higher computational performance of the binomial tree model at the same accuracy level is that its pricing is based on deterministic node recursion rather than random simulation and complex regression calculation, and only involves simple probability expectation and discount calculation, which is even more in line with the manual calculation habits of high school students [1, 4]. The LSMC method involves the generation of random paths and the performance of least squares regression for the in-the-money paths at each time node, which increases the calculation steps and time, but its random simulation technique does not confine it to the two kinds of asset price changes of the binomial tree model and has greater flexibility in real applications [2, 6].

4.2.2. Limitations of the study

There are certain limitations in the experimental design and research scope:

- a. The experimental parameters are relatively simple: only a set of standard parameters is selected for research, and the influence of different risk-free interest rates, asset price volatility and expiration dates on the pricing results of the two methods is not considered [4];
- b. The parameter setting range is small: the number of time steps of the binomial tree model and the number of simulation paths of the LSMC method are set in a small range, and the change rules of the two methods under larger parameters are not studied [1, 2];
- c. The calculation method is simple: the experiment is based on manual calculation, without using professional software for simulation and calculation, and the calculation results may have certain manual calculation errors [3, 5].

4.2.3. Applicable scene of the two methods

Combined with the experimental results and theoretical characteristics, the applicable scenes of the two methods are summarized for the basic application level [1, 2, 4]:

- a. The binomial tree model is suitable for the scenario of fast rough pricing with low requirements on pricing accuracy. Its principle is intuitive, the calculation steps are simple, and it can quickly obtain the approximate value of the American put option price with a small number of time steps, which is suitable for the initial judgment of option transactions;
- b. The LSMC method is suitable for the scenario of high-precision pricing with complex asset price changes. Although its calculation cost is higher, it does not limit the change form of asset price, and can simulate more complex asset price processes by increasing the number of simulation paths, so it has wider application prospects in the actual financial market.

5. Conclusion

This study focuses on American put options and uses the optimal stopping strategy as a key idea to explain the binomial tree model's principles and implementation steps in a simplified way. It also numerically compares the two methods in terms of pricing accuracy and calculation efficiency using the same set of standard parameters.

The main findings are summarized as follows:

First of all, the binomial tree model and LSMC method are both competent in solving the pricing problem of American. Besides, the accuracy of prices from both methods depends on their key parameters: the more time steps in the binomial tree model, the better the pricing accuracy; similarly, the more simulation paths in the LSMC method, the higher the pricing accuracy will be.

Secondly, the two methods are very different in terms of computational efficiency: the binomial tree model consists of clearly defined node recursion and calculations that are straightforward and quite efficient; therefore, this method is feasible for manual calculations. On the other hand, the LSMC method not only requires random simulation but also involves the least square regression step, making it less efficient when matched at the same level of accuracy.

Thirdly, the two methods are good at different things: the binomial tree model works swiftly and roughly to price options, while the LSMC method, due to its strong flexibility, can be used for high precision pricing of complex asset price paths.

This paper only explores the parameters in the simplest way. Later work can consider a wider range of parameters, more complex asset price models, and use professional software for simulation, which will bring the research results closer to real financial market applications.

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