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American Option Pricing and Early-Exercise Premium under the Binomial Tree Model

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Abstract: This paper discusses a comprehensive analysis of the pricing of two typed of options and early exercise premiums under the binomial tree model. We find the differences between European and American options, with particular emphasis on the conditions governing optimal early exercise strategies.We verify that early exercise is never the best option for American call options on stocks that pay no dividends, making their pricing the same as their European counterparts.Moreover, in the single-period model, the exercise of American put options depends on $K^* = \frac{uq}{r+q} S_0$, while in contrast, the relative positioning of the strike price and the ultimate stock value influences the exercise decisions in the two-period model.Our analysis indicates that the early exercise premium primarily manifests in multi-period—where holders can determine their own exercise timing—creating additional value relative to European put options.

Keywords: American options; European options; binomial tree model; early exercise premium; option pricing.

1. Introduction

As significant financial derivatives, options are crucial for investing and risk management. The two most basic option types are the American and European options. A key feature of American options is their exercisability at any point during the life of the contract, while the exercise of a European option is strictly restricted to the expiration date. The flexibility in exercising American options directly affects their pricing mechanisms and the value of practical application.

For special options like American options that allow early exercise, the binomial tree model enables more direct handling. However,little study has been done on the complexity caused by the early exercise feature of American options; instead, most of the existing research focuses on the price of European options. For example, in American options, when should early exercise be prioritized and what is the corresponding premium related to early exercise?

Against this backdrop, this paper will calculate the prices of the options using the binomial tree model. It will also specify the premium and the conditions under which an early exercise of an American option can optimize the holder's payout.

Specifically, this paper addresses the following questions: (1) Under what conditions does early exercise become optimal for American put options? (2) How does the early-exercise premium vary across different market parameters and model specifications? (3) What practical guidance can be provided to investors regarding exercise decisions? Our contribution lies in providing explicit analytical solutions for both single-period and two-period settings, complete with numerical implementations and practical insights for option traders.

2. Literature review

The research in this paper is grounded in the binomial tree model framework and aims to systematically analyze the pricing differences and early exercise premiums of American and European options. This review will sort out the related literature from three aspects: classical pricing theory, evolution and recent progress of numerical methods, and research on early exercise characteristics of American options.

2.1. The foundation and evolution of classical pricing theory

The European option pricing model (B-S model), published by Black[1] and Merton[2], is where the theoretical basis of the American option pricing can be found. It sets the foundation for all subsequent option studies.However, the B-S model cannot directly address the issue of whether American options are exercised early, prompting a substantial body of literature on numerical methods. Among them, the Binomial Options Pricing Model (BOPM) proposed by Cox, John C and Ross[3] is a fundamental and intuitive framework to deal with early strike premiums.

As the number of steps approaches infinity, the CRR model not only derives the B-S formula in an intuitive manner, but more importantly, this model introduces the principle of risk neutrality, comparing the value of immediate exercise with the value of holding at each node, thereby naturally incorporating the early exercise decision-making process into the pricing mechanism.This core idea became the basis for all subsequent numerical pricing methods based on lattices (Lattice Methods) and finite differences (Finite Difference Methods).

2.2. Refinement and recent advances in numerical computing methods

After the CRR model, Broadie[4] conducted a systematic review and comparison of various numerical methods. Due to its forward-looking nature, conventional Monte Carlo simulation is still insufficient when it comes to tackling the early exercise problem associated with American options. Nevertheless, this problem is successfully addressed via the least squares Monte Carlo (LSM) simulation method put forward by Longstaff[5].

In recent years, emerging computational methods represented by machine learning are being widely applied to American option pricing. Ruf[6] proposes a neural network-based approach that transforms the pricing problem into a stochastic control problem without pre-specifying the underlying asset dynamics, making the computation more flexible. Becker[7], on the other hand, delves into the performance of deep learning, especially deep neural networks, in American option pricing, an approach that has more potential when dealing with high-dimensional problems. Batchuluun[8]'s research applies machine learning to markets with transaction costs, further confirming the effectiveness of machine learning methods in pricing complex derivatives. Meanwhile, traditional numerical

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methods continue to be optimised. Lo[9] focuses on solving high-dimensional challenges in American option pricing and develops a high-performance algorithm based on tensor trees, which significantly improves computational speed.

2.3. In-depth Research on the early execution characteristics of American options

Within the theoretical research on the characteristics of American options, a classical finding holds that early execution is never a optimal course of action for American call options on stocks that do not distribute dividends. As a result, the pricing of such American call options coincides with that of European call options. The core of the study focuses on American put options. In his paper, Cox[3] explicitly states that the possibility of early execution brings a premium to it. In a follow-up study, Geske[10] analyzed the option value of the right to early execution itself through a composite option model. Kim[11], on the other hand, obtained an analytic integral expression for the price of an American put option. In recent years, Li[12] provides a full summary of the finite difference method in American options pricing and discusses the treatment of boundary conditions.

3. Discussion

This paper will discuss the pricing mechanisms along with the conditions governing their exercise, by leveraging both single-period and two-period binomial tree frameworks.Below are the interpretations of the mathematical notations used throughout the proofs.

Table 1: Mathematical notation

Symbol	Description
S_0	initial stock price
K	strike price
u	upside factor ($u > 1$)
d	downside factor ($0 < d < 1$)
r	risk-free interest rate ($r > 0$)
q	risk-neutral probability: $q = \dfrac{1+r-d}{u-d}$

3.1. Pricing and exercise conditions under a single-period binomial tree model

The single-period binomial tree model stands as the most fundamental option pricing model in financial engineering. It posits that, over the option's lifetime, the price of the underlying asset can follow only two potential trajectories of change: an upward movement to a certain price or a downward movement to a certain price.

3.1.1. Pricing of European options

The exercise of European options is restricted exclusively to the time point $t = 1$.

At maturity ($t = 1$), the payoffs of European **call options** with strike K are

$$(S_0u - K)^+ \quad \text{and} \quad (S_0d - K)^+,$$

corresponding to the up and down states.

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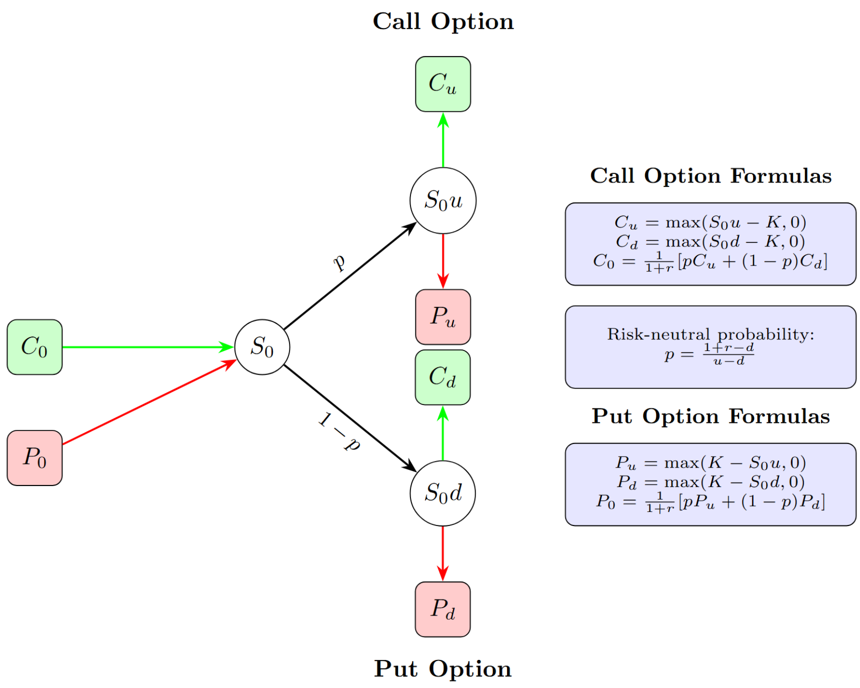

Figure 1: single-period binomial tree model

Under the risk-neutral pricing framework:

$$C_E = \frac{1}{1+r}\left[q \cdot (S_0u - K)^+ + (1-q) \cdot (S_0d - K)^+\right] \tag{1}$$

For European **put options**, the formula is derived as follows:

$$P_E = \frac{1}{1+r}\left[q \cdot (K - S_0u)^+ + (1-q) \cdot (K - S_0d)^+\right] \tag{2}$$

3.1.2. The General recursive concept of American options

American options permit exercise immediately at $t = 0$ or holding until $t = 1$ before exercising. Consequently, their value is the greater of the two alternatives.

For the **American call option**, the holder faces the following decision at $t = 0$:

- Immediate exercise: $(S_0 - K)^+$,
- Continue holding: C_E.

Thus we obtain

$$C_A = \max\left((S_0 - K)^+, C_E\right). \tag{3}$$

Similarly, for the **American put option**, the holder also chooses between immediate exercise and continuation:

- Immediate exercise: $(K - S_0)^+$,
- Continue holding: P_E.

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Hence

$$P_A = \max\big((K - S_0)^+,\ P_E\big). \tag{4}$$

These recursive definitions provide the general framework for evaluating American options. In subsequent sections, we shall progressively develop the algebraic proofs and identify the critical conditions for both call and put options, under scenarios with and without a single discrete dividend.

3.1.3. American call options on a non-dividend-paying stock

There exists a well-established conclusion regarding this situation, and we shall proceed to demonstrate it.

$$C_A = C_E$$

Proof. We prove $(S_0 - K)^+ \leq C_E$ in all cases.

Case 1: $S_0 \leq K$

When $S_0 \leq K$,while$(S_0 - K)^+ = \max(S_0 - K, 0) = 0$.Non-positive exercise gain due to asset price falling short of exercise price.

Since option prices are non-negativity,$C_E \geq 0$.Thus:

$$(S_0 - K)^+ \leq C_E$$

Case 2: $S_0 > K$

$$(S_0 u - K)^+ \geq S_0 u - K$$
$$(S_0 d - K)^+ \geq S_0 d - K$$

Let us define the discount factor as $\beta = \frac{1}{1+r}$. Substituting into C_E definition:

$$C_E \geq \beta\left[q(S_0 u - K) + (1-q)(S_0 d - K)\right]$$

Simplify right-hand side,then we can obtain:

$$C_E \geq \beta(S_0(1+r) - K) = S_0 - \beta K$$

Because $r > 0$, the discount factor β is less than 1. It follows directly that:

$$S_0 - \beta K > S_0 - K.$$

Therefore:

$$C_E > S_0 - K = (S_0 - K)^+$$

From both cases:

$$(S_0 - K)^+ \leq C_E$$

$$C_A = \max\left(\underbrace{(S_0 - K)^+}_{\leq C_E},\ \underbrace{C_E}_{=C_E}\right) = C_E$$

In a summary,for non-dividend stocks in single-period binomial model:

$$C_A = C_E \tag{5}$$

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3.1.4. The pricing and exercise conditions of American put options on a non-dividend-paying stock

We have ascertained that the pricing formula for American put options is

$$P_A = \max\big((K - S_0)^+,\, P_E\big). \tag{6}$$

Therefore, to determine when immediate exercise is optimal, we need to compare

$$(K - S_0)^+ \;\geq\; \beta\left[q(K - S_0u)^+ + (1-q)(K - S_0d)^+\right].$$

Since it contains the positive modular function $(\cdot)^+$, we split into the three three cases.

Case 1: $S_0 \geq K$:
Then $(K - S_0)^+ = 0$ while $P_E \geq 0$. Therefore, it is preferable to continue holding $P_A = P_E$.

Case 2: $K \geq S_0u$:
Both states are in-the-money:

$$(K - S_0u)^+ = K - S_0u, \qquad (K - S_0d)^+ = K - S_0d.$$

Thus

$$P_E = \beta\big[q(K - S_0u) + (1-q)(K - S_0d)\big].$$

Simplify:

$$q(K - S_0u) + (1-q)(K - S_0d) = K - S_0(qu + (1-q)d).$$

At this point we use the following identity:

$$qu + (1-q)d = 1/\beta = 1 + r \tag{7}$$

Proof of identity. We know that

$$S_0 = \frac{1}{1+r}\big(qS_0u + (1-q)S_0d\big).$$

Dividing both sides by S_0 yields

$$1 = \frac{1}{1+r}\big(qu + (1-q)d\big),$$

Substituting into the identity (7),we obtain

$$P_E = \beta K - S_0.$$

Compared with immediate exercise:

$$(K - S_0) - P_E = K(1-\beta) > 0,$$

so immediate exercise strictly dominates, $P_A = K - S_0$.

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Case 3: $S_0 < K < S_0 u$**:**
Up state out-of-the-money, down state in-the-money:

$$(K - S_0 u)^+ = 0, \qquad (K - S_0 d)^+ = K - S_0 d.$$

Therefore

$$P_E = \frac{1-q}{1+r}(K - S_0 d).$$

Immediate exercise is optimal if

$$K - S_0 \ \geq \ \frac{1-q}{1+r}(K - S_0 d).$$

Multiply by $(1+r)$, then expand and rearrange:

$$K(r+q) \ \geq \ S_0\big[(1+r) - (1-q)d\big].$$

Coefficients:

$$(1+r) - (1-q)d = qu.$$

Thus

$$K(r+q) \ \geq \ S_0 qu.$$

The critical strike is therefore

$$K^* = \frac{uq}{r+q} S_0.$$

Conclusion (no dividend, single period):

- If $K \geq K^*$: immediate exercise is optimal, $P_A = K - S_0$;
- If $K < K^*$: continuation is optimal, $P_A = P_E$.

Economic interpretation: K^* is the critical strike price at which the immediate exercise payoff equals the holding payoff.Higher K or larger r increases the likelihood of early exercise.

3.1.5. Effect of a single discrete dividend D paid at $t \in (0, 1)$

Suppose a known cash dividend $D > 0$ is paid during $(0, 1)$. The terminal stock price shall be adjusted to:

$$S_0 u - D, \qquad S_0 d - D.$$

(i) European option valuation:

$$C_E = \beta\Big[q((S_0 u - D) - K)^+ + (1-q)((S_0 d - D) - K)^+\Big],$$

$$P_E = \beta\Big[q(K - (S_0 u - D))^+ + (1-q)(K - (S_0 d - D))^+\Big].$$

(ii) American call options: dividends may be subject to early exercise..
Immediate exercise: $(S_0 - K)^+$. Continuation: C_E.
Early exercise condition:

$$(S_0 - K)^+ \ \geq \ C_E.$$

Assume that $S_0 > K$, and that both the up and down movements of the S_0 in the subsequent period, the price remains above the strike price.Under these conditions, the C_E may be approximated by:

$$C_E = S_0 - \beta K - \beta D.$$

Here:

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- βK is the discounted exercise price;
- βD is the present value of the dividend to be paid in the next period, which cannot be captured if the option is held instead of immediately exercised.

The immediate exercise value is $(S_0 - K)$, while the European option value is C_E. The difference between the two is:

$$\begin{aligned}(S_0 - K) - C_E &= (S_0 - K) - (S_0 - \beta K - \beta D)\\ &= -K + \beta K + \beta D\\ &= \beta(D - rK).\end{aligned}$$

Immediate exercise is optimal if:

$$(S_0 - K) - C_E \geq 0 \quad \Longrightarrow \quad \beta(D - rK) \geq 0 \quad \Longrightarrow \quad D \geq rK,$$

i.e., immediate exercise is preferred when the dividend D exceeds the interest on the exercise price rK:

$$D > rK.$$

(iii) American put options: dividends may diminish the incentive for early exercise.
Immediate exercise: $(K - S_0)^+$. Continuation: P_E.
Early exercise inequality:

$$(K - S_0)^+ \ \geq \ P_E.$$

Case 1: both states in-the-money$(S_0 < K)$
Thus, if both the up and down states after dividend remain below the strike, i.e.,

$$K \ \geq \ S_0 u - D,$$

then the continuation value is

$$P_E = \beta K - S_0 + \beta D,$$

and

$$(K - S_0) - P_E = \beta(rK - D).$$

Hence, immediate exercise is optimal if

$$K > \tfrac{D}{r}.$$

Case 2: only the down state in-the-money.
That is, the stock is currently above K but may fall below it after the dividend, i.e.,

$$S_0 < K < S_0 u - D.$$

Then the continuation value is

$$P_E = \frac{1-q}{1+r}\,(K - S_0 d + D).$$

The critical condition for early exercise becomes

$$K \ \geq \ \frac{quS_0 + (1-q)D}{r+q}.$$

Compared to the critical level $K^* = \frac{uq}{r+q}S_0$ when no dividend is paid, the presence of a dividend $D > 0$ raises the critical strike price, thereby reducing the incentive for early exercise.

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3.2. Pricing and exercise conditions under a two-period binomial tree model

Compared to single-period framework, the two-period binomial tree framework more fully captures the dynamic characteristics of option pricing—the decision to exercise an American option early may take place not only at the initial time but also at any intermediate node ($t = 1$) in the pricing tree.This setting lends greater practical relevance to analyses of optimal exercise strategies and associated early exercise premiums.

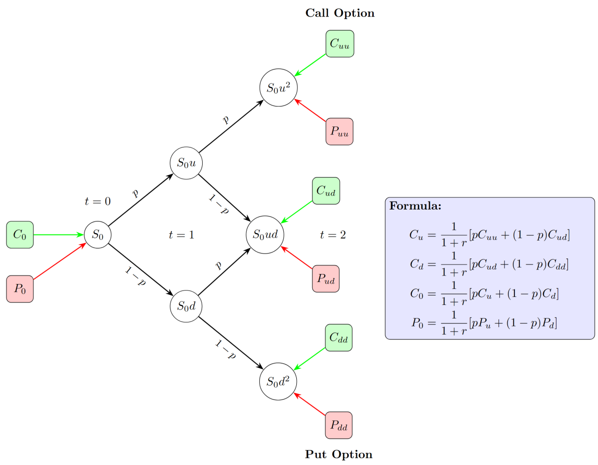

Figure 2: The two-period binomial tree model

3.2.1. Pricing of European options

(i) Without dividends

Following the same lines as in the single period, assuming no dividends, three possible prices may occur at maturity:

$$S_{uu} = S_0u^2, \qquad S_{ud} = S_0ud, \qquad S_{dd} = S_0d^2.$$

Due to the previous definition of β

$$C_E^{(0)} = \beta^2\Big[q^2(S_{uu} - K)^+ + 2q(1-q)(S_{ud} - K)^+ + (1-q)^2(S_{dd} - K)^+\Big], \tag{8}$$

$$P_E^{(0)} = \beta^2\Big[q^2(K - S_{uu})^+ + 2q(1-q)(K - S_{ud})^+ + (1-q)^2(K - S_{dd})^+\Big]. \tag{9}$$

This corresponds is identical to the single-period scenario.

(ii) With dividends

If a discrete dividend D is distributed in either the first or second period, the stock price must be adjusted to its ex-dividend value. For instance, if a dividend D is paid at the conclusion of period 1, then:

$$S_{uu} = S_0u^2 - Du, \qquad S_{ud} = S_0ud - Dd, \qquad S_{dd} = S_0d^2 - Dd.$$

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The corresponding value of the European option is

$$C_E^{(D)} = \beta^2\Big[q^2(S_{uu}-K)^+ + 2q(1-q)(S_{ud}-K)^+ + (1-q)^2(S_{dd}-K)^+\Big], \tag{10}$$

$$P_E^{(D)} = \beta^2\Big[q^2(K-S_{uu})^+ + 2q(1-q)(K-S_{ud})^+ + (1-q)^2(K-S_{dd})^+\Big]. \tag{11}$$

As dividends reduce the future share price level, therefore:

- For call options, holding value diminishes and the option's value decreases;
- For put options, the lower the future share price, the greater the value.

3.2.2. American call options on a non-dividend-paying stock

In such case, the holder evaluates, at each node in the decision tree, the trade-off between exercising the option immediately and continuing to hold it.

At maturity $t = 2$, the option values are simply the payoffs:

$$V_2(S_{uu}) = (S_{uu}-K)^+, \quad V_2(S_{ud}) = (S_{ud}-K)^+, \quad V_2(S_{dd}) = (S_{dd}-K)^+.$$

At time $t = 1$

$$S_u = S_0 u, \qquad S_d = S_0 d.$$

$$V_1(S_u) = \max\Big\{(S_u-K)^+,\ \frac{1}{1+r}\big[qV_2(S_{uu}) + (1-q)V_2(S_{ud})\big]\Big\},$$

$$V_1(S_d) = \max\Big\{(S_d-K)^+,\ \frac{1}{1+r}\big[qV_2(S_{ud}) + (1-q)V_2(S_{dd})\big]\Big\}.$$

At the initial time $t = 0$

$$C_A = \max\Big\{(S_0-K)^+,\ \frac{1}{1+r}[qV_1(S_u) + (1-q)V_1(S_d)]\Big\}. \tag{12}$$

Result: For a stock with no dividend payments:

$$C_A = C_E^{(0)}, \tag{13}$$

and early exercise never occurs.

3.2.3. The pricing and exercise conditions of American put options on a non-dividend-paying stock

For an American put option, early exercise may be optimal. The values at $t = 1$ are

$$V_1(S_u) = \max\Big\{(K-S_u)^+,\ \frac{1}{1+r}[qV_2(S_{uu}) + (1-q)V_2(S_{ud})]\Big\},$$

$$V_1(S_d) = \max\Big\{(K-S_d)^+,\ \frac{1}{1+r}[qV_2(S_{ud}) + (1-q)V_2(S_{dd})]\Big\}.$$

At $t = 0$

$$P_A = \max\Big\{(K-S_0)^+,\ \frac{1}{1+r}[qV_1(S_u) + (1-q)V_1(S_d)]\Big\}. \tag{14}$$

r can be understood as the value of time. If r is a negative interest rate, it means that the funds held will depreciate. This does not conform to the financial market in real life. Therefore, r must be a

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positive number.That r is greater than 0 is also more in line with people's behavioral logic and market rules.

$$0 < r < 1 \tag{15}$$

$$\frac{1}{1+r} < 1 \tag{16}$$

Exercise conditions. In single-period analysis, we know that if the price is S, the condition under which immediate exercise is preferable to holding is

$$K \geq \frac{uq}{r+q}S.$$

Substituting into $S_u = S_0 u, S_d = S_0 d:$

$$K_u^* = \frac{u^2 q}{r+q}S_0, \qquad K_d^* = \frac{uqd}{r+q}S_0,$$

with $K_u^* > K_d^*$. So under the two-period binomial tree:

- If $K \geq K_u^*$, then immediate exercise is optimal at all nodes.
- If $K < K_d^*$, then continuation is optimal at all nodes.
- If the strike price lies between the two thresholds, i.e. $K_d^* \leq K < K_u^*$, then a mixed situation arises: one node exercises its rights, while another node continues to hold.

Case 1: $K \geq K_u^*$ (exercise at both S_u and S_d).
At $t = 1$:

$$V_1(S_u) = K - S_u, \qquad V_1(S_d) = K - S_d.$$

At $t = 0$:

$$C_0 = \beta[q(K - S_u) + (1-q)(K - S_d)] = \beta K - S_0.$$

Compare $K - S_0$ with C_0

$$(K - S_0) - C_0 = \frac{r}{1+r}K > 0,$$

immediate exercise at $t = 0$ is optimal:

$$P_A = K - S_0. \tag{17}$$

Case 2: $K_d^* \leq K < K_u^*$ (exercise at S_d, continuation at S_u).
At $t = 1$, the option values are

$$V_1(S_u) = \beta[qV_2(S_{uu}) + (1-q)V_2(S_{ud})], \qquad V_1(S_d) = K - S_d.$$

Thus at $t = 0$ the holding value is

$$C_0 = \beta\big[qV_1(S_u) + (1-q)(K - S_d)\big]. \tag{18}$$

We need to compare $K - S_0$ with C_0. Since $V_1(S_u)$ depends on the payoffs $V_2(\cdot)$, we categorise K and the relative positions of the three terminal prices into four subcases(A–D).

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Subcase A: $K \leq S_{dd}$ (no payoff at maturity).

For any path,$Sij \geq Sdd \geq K$.Thus all terminal payoffs vanish:

$$V_2(S_{uu}) = V_2(S_{ud}) = V_2(S_{dd}) = 0.$$

Thus $V_1(S_u) = 0$, $V_1(S_d) = K - S_d$, and by (18),

$$C_0 = \beta(1-q)(K - S_d).$$

Comparing $K - S_0$ with C_0, and multiplying by $(1+r)$,

$$(1+r)(K - S_0) \geq (1-q)(K - S_d).$$

Rearranging yields

$$K(r+q) \geq S_0 qu,$$

We obtain

$$K \geq \frac{qu}{r+q} S_0. \tag{19}$$

Hence immediate exercise at $t = 0$ is optimal if (19) holds.

Subcase B: $S_{dd} < K \leq S_{ud}$ (only the lowest node yields payoff).

Here $V_2(S_{dd}) = K - S_{dd} > 0$, while $V_2(S_{ud}) = V_2(S_{uu}) = 0$. Thus $V_1(S_u) = 0$, $V_1(S_d) = K - S_d$, and C_0 is identical to subcase A.

$$C_0 = \beta(1-q)(K - S_d),$$

Similarly obtained

$$K \geq \frac{qu}{r+q} S_0 \tag{20}$$

as $t = 0$ exercise condition.

Subcase C: $S_{ud} < K \leq S_{uu}$ (two lower nodes yield payoff).

Here

$$V_2(S_{uu}) = 0, \quad V_2(S_{ud}) = K - S_{ud} > 0, \quad V_2(S_{dd}) = K - S_{dd} > 0.$$

Therefore

$$V_1(S_u) = \beta(1-q)(K - S_{ud}), \qquad V_1(S_d) = K - S_d.$$

From (18),

$$C_0 = \beta \left[q \cdot \beta(1-q)(K - S_{ud}) + (1-q)(K - S_d) \right].$$

Multiplying both sides by $\beta^2 > 0$, the condition $K - S_0 \geq C_0$ becomes

$$\left(\frac{1}{\beta}\right)^2 (K - S_0) \ \geq \ q(1-q)(K - S_{ud}) + \frac{1}{\beta}(1-q)(K - S_d). \tag{21}$$

Expanding both sides of (21): Left-hand side:

$$\left(\frac{1}{\beta}\right)^2 K - \left(\frac{1}{\beta}\right)^2 S_0.$$

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And right-hand side:

$$q(1-q)K - q(1-q)S_{ud} + \frac{1}{\beta}(1-q)K - \frac{1}{\beta}(1-q)S_d.$$

Collecting terms gives

$$K\Big[\Big(\frac{1}{\beta}\Big)^2 - q(1-q) - \frac{1}{\beta}(1-q)\Big] \;\geq\; S_0\Big[\Big(\frac{1}{\beta}\Big)^2 - q(1-q)ud - \frac{1}{\beta}(1-q)d\Big]. \tag{22}$$

Simplifying the coefficient of K and S_0:

$$K(r^2+r+rq+q^2) \;\geq\; q^2u^2S_0,$$

We obtain:

$$K \geq \frac{q^2u^2}{r^2+r+rq+q^2}\,S_0. \tag{23}$$

If the inequality holds, exercise at $t=0$; otherwise, continue holding.

Subcase D: $K > S_{uu}$ **(all nodes yield payoff).**

Here

$$V_2(S_{uu}) = K - S_{uu}, \quad V_2(S_{ud}) = K - S_{ud}, \quad V_2(S_{dd}) = K - S_{dd}.$$

Then

$$V_1(S_u) = \beta[q(K-S_{uu}) + (1-q)(K-S_{ud})],$$
$$V_1(S_d) = \beta[q(K-S_{ud}) + (1-q)(K-S_{dd})].$$

Substituting into (18) and simplifying yields

$$C_0 = \beta(K) - S_0.$$

Comparing with immediate exercise,

$$(K-S_0) - C_0 = \beta(r)K > 0,$$

so immediate exercise at $t=0$:

$$P_A = K - S_0. \tag{24}$$

Case 3: $K_d^* \leq K < K_u^*$ **(exercise at** S_u**, continuation at** S_d**).**

Symmetrical to Case 2, when the option chooses immediate exercise at the upper node Su and continues holding at the lower node Sd, analogous early exercise conditions can be derived. The detailed step-by-step derivation is presented in Appendix 4.; the main text provides only the summary as follows.

Case 4: $K < K_d^*$ **(never exercise early).**

At $t=1$, the value of holding continues to be dominant at both nodes:

$$V_1(S_u) = \beta[qV_2(S_{uu}) + (1-q)V_2(S_{ud})], \qquad V_1(S_d) = \beta[qV_2(S_{ud}) + (1-q)V_2(S_{dd})].$$

At $t=0$ this yields

$$C_0 = \beta^2\big[q^2V_2(S_{uu}) + 2q(1-q)V_2(S_{ud}) + (1-q)^2V_2(S_{dd})\big].$$

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Table 2: Exercise conditions for Case 3 (upper node exercise, lower node continuation).

Subcase	**Threshold condition at $t = 0$**
A: $K \le S_{dd}$	$K \ge \dfrac{(1-q)d}{r+1-q}\,S_0$
B: $S_{dd} < K \le S_{ud}$	$K \ge \dfrac{q(1+r)u + (1-q)^2 d^2}{(1+r)^2 - q(1+r) - (1-q)^2}\,S_0$
C: $S_{ud} < K \le S_{uu}$	$K \ge \dfrac{q(1+r)u + (1-q)\big[q\,ud + (1-q)d^2\big]}{(1+r)^2 - q(1+r) - (1-q)^2}\,S_0$
D: $K > S_{uu}$	$P_A = K - S_0$ (immediate exercise)

Remark: Since K is relatively small, at least some $(K-S)^+$ terms at maturity will vanish.For example, $(K - S_{uu})^+ = 0$ if $S_{uu} > K$.Therefore

$$P_A = P_E.$$

Summary:

Table 3 summarizes the four possible cases.

- In **Case 1**, immediate exercise dominates at both nodes, so the American put price is precisely equal to its intrinsic value $K - S_0$.
- In **Case 2**, continuation is optimal at the upper node while immediate exercise is optimal at the lower node. Depending on the strike K relative to the terminal stock prices (S_{uu}, S_{ud}, S_{dd}), different inequalities arise, capturing when early exercise at $t = 0$ is preferable.
- In **Case 3**, the situation is reversed: exercise at the upper node and continuation at the lower node. Again, explicit strike thresholds determine whether immediate exercise at $t = 0$ is optimal.
- Finally, in **Case 4**, continuation dominates at both nodes, so the American put coincides with the European put, $P_A = P_E$, and no early exercise premium exists.

3.3. Early exercise premium

The early exercise premium represents the difference between the American and European put option prices:

$$\text{EEP} = P_A - P_E \;\ge\; 0. \tag{25}$$

It measures the additional value gained by investors due to their right to exercise options early. In this section, we summarize the results for both the one-period and two-period binomial tree models.

3.3.1. One-period model

For the one-period case, the option values at $t = 0$ are

$$P_E = \beta\big[q(K - S_u)^+ + (1-q)(K - S_d)^+\big], \qquad P_A = \max\{K - S_0, P_E\}.$$

By comparing immediate exercise with continuation, the early exercise condition is

$$K \ge \frac{uq}{r+q} S_0.$$

- If K satisfies the above inequality, then $P_A > K - S_0 \ge P_E$, and $\text{EEP} > 0$.
- Otherwise, $P_A = P_E$, hence $\text{EEP} = 0$.

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Table 3: Summary of exercise conditions for Cases 1–4 (two-period binomial tree)

Case	Description	Optimal exercise rule at $t = 0$
Case 1	Exercise at both nodes (S_u, S_d)	Immediate exercise: $P_A = K - S_0$
Case 2	Continuation at S_u, exercise at S_d	Depend on K: A: $K \le S_{dd}$, $K \ge \dfrac{qu}{r+q} S_0$ B: $S_{dd} < K \le S_{ud}$, $K \ge \dfrac{qu}{r+q} S_0$ C: $S_{ud} < K \le S_{uu}$, $K \ge \dfrac{q^2u^2}{r^2+r+rq+q^2} S_0$ D: $K > S_{uu}$, immediate exercise: $P_A = K - S_0$
Case 3	Exercise at S_u, continuation at S_d	Depend on K: A: $K \le S_{dd}$, $K \ge \dfrac{(1-q)d}{r+1-q} S_0$ B: $S_{dd} < K \le S_{ud}$, $K \ge \dfrac{q(1+r)u + (1-q)^2d^2}{(1+r)^2 - q(1+r) - (1-q)^2} S_0$ C: $S_{ud} < K \le S_{uu}$, $K \ge \dfrac{q(1+r)u + (1-q)\left[q\,ud + (1-q)d^2\right]}{(1+r)^2 - q(1+r) - (1-q)^2} S_0$ D: $K > S_{uu}$, immediate exercise: $P_A = K - S_0$
Case 4	Continuation at both nodes (S_u, S_d)	No early exercise: $P_A = P_E$

3.3.2. Two-period model

The analysis in Sections 3.2.2 and 3.2.3 shows how the early exercise premium depends on the strike K relative to the possible terminal stock prices:

- **Case 1:** Immediate exercise dominates at both nodes. Here $P_A = K - S_0$ and $P_E < P_A$, hence $\text{EEP} > 0$.
- **Case 2 and Case 3:** Mixed exercise/continuation decisions at $t = 1$. In these intermediate cases, the strike thresholds derived in Subcases A–C determine whether early exercise at $t = 0$ is optimal. When early exercise occurs, $P_A > P_E$ and the early exercise premium is strictly positive. Otherwise, $P_A = P_E$.
- **Case 4:** Continuation dominates at both nodes. In this case $P_A = P_E$, hence $\text{EEP} = 0$.

The early exercise premium reflects the trade-off between the immediate exercise gain $(K - S)^+$ and the value of holding the asset over time, the latter being calculated through Arrow-Debreu weighted expectations to determine future returns.

3.4. Python implementation

To demonstrate the calculation of American option prices under a binary tree, we provide a straightforward Python implementation covering both the one-period and two-period cases. In practice,the method can be generalized to accommodate any finite number of periods.

It should be emphasized that our Python implementation focuses exclusively on *American put options on stocks that pay no dividends*. American call options are not implemented, and dividends are not considered, for the following reasons:

1. **American call options without dividends never exercise early.**
2. **The early exercise premium arises mainly in American puts.** For put options, when K is high and S_0 is low, immediate exercise may be preferable to holding the position, thereby yielding a

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positive early exercise premium (EEP). Therefore, American put options are the primary focus for early exercise analysis.

3. **Dividends are excluded for simplicity.** If dividends are present, American call options may indeed be exercised early to capture dividend payments. However, modeling dividends requires additional adjustments to the stock price dynamics. Given that this study aims to examine the outcomes of early exercise versus early exercise premium under the binomial tree framework,we restrict attention to the no-dividend case for clarity.

This algorithm is based on backward induction, starting from the terminal node's payoff, it progressively traces back.At each node, comparing $(K-S)^+$ with the continuation value.The value at the root node constitutes the American option price.

Algorithm 1 American put options valuation under a binomial tree

Input: Initial S_0, K, u, d, r, number of steps N.
Output: The price of the American put option $V(S_0)$.

```
Compute q = (1 + r − d)/(u − d).
Generate terminal stock prices S_ij and payoffs (K − S_ij)^+.
for t = N − 1 down to 0 do
    for each node S at time t do
        C = 1/(1 + r) (q V_up + (1 − q) V_down)
        Set option value V = max{K − S, C}.
    end for
end for
return V(S_0).
```

3.4.1. Numerical example

To demonstrate this algorithm, we set the following parameters:

$$S_0 = 100, \quad K = 100, \quad u = 1.1, \quad d = 0.9, \quad r = 0.05.$$

3.4.2. One-period case

The terminal stock prices are $S_u = 110$, $S_d = 90$. Then

$$P_E^{(1)} = \frac{1}{1.05}[q(0) + (1-q)(10)] \approx 2.38, \quad P_A^{(1)} = \max\{0, 2.38\} = 2.38.$$

Thus $\text{EEP}^{(1)} = 0$.

3.4.3. Two-period case

The terminal stock prices are $S_{uu} = 121$, $S_{ud} = 99$, $S_{dd} = 81$. The terminal payoffs are $(K-S_{uu})^+ = 0$, $(K-S_{ud})^+ = 1$, $(K-S_{dd})^+ = 19$. Backward induction yields

$$V_1(S_u) = \max\{0, \tfrac{1}{1.05}[q(0) + (1-q)(1)]\} \approx 0.238,$$

$$V_1(S_d) = \max\{10, \tfrac{1}{1.05}[q(1) + (1-q)(19)]\} \approx 10.$$

At the root,

$$C_0 = \tfrac{1}{1.05}[q \cdot 0.45 + (1-q) \cdot 10] \approx 2.551, \quad P_A^{(2)} = \max\{0, 2.551\} = 2.551.$$

The corresponding European option price is

$$P_E^{(2)} = \tfrac{1}{(1.05)^2}[q^2 \cdot 0 + 2q(1-q) \cdot 1 + (1-q)^2 \cdot 19] \approx 1.417.$$

Thus the early exercise premium is

$$\mathrm{EEP}^{(2)} = P_A^{(2)} - P_E^{(2)} \approx 1.134.$$

4. Conclusion

In this paper, we have conducted research in the following areas.

First, we demonstrate the classical result about American call options on stocks with no dividend payments, implying that their price coincides with that of European call options, irrespective of the number of periods in the model. This fundamental property arises from the time value of money and the opportunity cost associated with exercising the option prior to maturity.Second, we establish precise mathematical conditions determining the optimal exercise strategy on American put options. In single-period models, we derive the critical exercise threshold $K^* = \frac{uq}{r+q} S_0$ that determines whether immediate exercise is preferable to holding the position. More significantly, within a two-period framework, we fully categorise optimal strategies into four distinct exercise patterns based on the relationship between the strike price and the terminal stock price. This classification system assists investors in discerning when to exercise early versus maintaining the option position.Thirdly, numerical analysis quantifies the Early Exercise Premium (EEP) distinguishing American and European put options. The magnitude of EEP critically depends on the degree of in-the-money status, volatility parameters, and time to maturity.This paper also demonstrates how to calculate American and European option prices using Python.

This paper provides a structured analytical framework for understanding American option exercise decisions. Future research may extend the scope of analysis to incorporate dividend payments, stochastic volatility, transaction costs, and other market frictions that affect real-world exercise decisions. Furthermore, empirical validation of theoretical predictions under varying market conditions would further enhance the practical relevance of the findings.

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Appendix

In this appendix we provide the full step-by-step derivations for Case 3 $K_d^* \le K < K_u^*$ (exercise at S_u, continuation at S_d),corresponding to the summary in Table 2. The structure follows exactly the four subcases (A–D) as in Case 2.
In this case, the option values at $t = 1$ are

$$V_1(S_u) = K - S_u, \qquad V_1(S_d) = \beta[qV_2(S_{ud}) + (1-q)V_2(S_{dd})].$$

Thus at $t = 0$ the continuation value is

$$C_0 = \beta\big[q(K - S_u) + (1-q)V_1(S_d)\big]. \tag{26}$$

Subcase A: $K \le S_{dd}$ (no payoff at maturity).

All terminal payoffs vanish:

$$V_2(S_{uu}) = V_2(S_{ud}) = V_2(S_{dd}) = 0.$$

Hence $V_1(S_d) = 0$ and

$$C_0 = \beta q(K - S_u).$$

Comparing with immediate exercise $K - S_0$,

$$\frac{1}{\beta}(K - S_0) \;\ge\; q(K - S_u).$$

Rearranging,and since the risk-neutral identity, this condition reduces to

$$K \ge \frac{(1-q)d}{r + 1 - q}\, S_0. \tag{27}$$

Subcase B: $S_{dd} < K \le S_{ud}$ (only the lowest node yields payoff).

Here $V_2(S_{dd}) = K - S_{dd} > 0$, while $V_2(S_{ud}) = 0$. Thus

$$V_1(S_d) = \beta(1-q)(K - S_{dd}).$$

From (26),

$$C_0 = \beta\Big[q(K - S_u) + (1-q)\cdot\frac{1-q}{1+r}(K - S_{dd})\Big].$$

Multiplying both sides by $(1+r)^2$, then substituting $S_u = S_0u$, $S_{dd} = S_0d^2$ and rearranging yields

$$K \;\ge\; \frac{q(1+r)u + (1-q)^2d^2}{(1+r)^2 - q(1+r) - (1-q)^2}\, S_0. \tag{28}$$

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Subcase C: $S_{ud} < K \leq S_{uu}$ (two lowest nodes yield payoff).

Here

$$V_2(S_{ud}) = K - S_{ud} > 0, \quad V_2(S_{dd}) = K - S_{dd} > 0.$$

Thus

$$V_1(S_d) = \beta\big[q(K - S_{ud}) + (1-q)(K - S_{dd})\big].$$

Substituting into (26):

$$C_0 = \beta\left[q(K - S_u) + (1-q)\cdot\beta\big(q(K - S_{ud}) + (1-q)(K - S_{dd})\big)\right].$$

Multiplying by $(1+r)^2$ yields and substituting $S_u = S_0 u,\ S_{ud} = S_0 ud,\ S_{dd} = S_0 d^2$, the inequality becomes

$$K \ \geq \ \frac{q(1+r)u + (1-q)\big[q\,ud + (1-q)d^2\big]}{(1+r)^2 - q(1+r) - (1-q)^2}\,S_0. \tag{29}$$

Subcase D: $K > S_{uu}$ (all nodes yield payoff).

Here

$$V_2(S_{ud}) = K - S_{ud}, \quad V_2(S_{dd}) = K - S_{dd}.$$

Hence

$$V_1(S_d) = \beta[q(K - S_{ud}) + (1-q)(K - S_{dd})].$$

Substituting into (26) yields

$$C_0 = \beta[q(K - S_u) + (1-q)V_1(S_d)].$$

Direct calculation shows that

$$C_0 = \beta K - S_0,$$

so that

$$(K - S_0) - C_0 = \beta K > 0,$$

implying immediate exercise at $t = 0$ is optimal:

$$P_A = K - S_0. \tag{30}$$