

A Comprehensive Model of Stock Prices: The Role of Bonds, Stocks, and Adaptive Expectations

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Abstract: This research develops a straightforward financial market model featuring a single bond and a single stock, each governed by basic pricing formulas. By applying techniques for solving differential equations, the paper demonstrates that under very mild assumptions—such as the absence of arbitrage opportunities and rational investor behavior—the stock price converges to its fundamental value over time. These assumptions ensure that expected returns from the bond and the stock align, illustrating how stock prices adjust in response to market dynamics. The focus is on the application of differential equations to show how the stock price moves toward its intrinsic value. It also provides insights into investor's behavior and market adjustments.

Keywords: finance, asset pricing, rational expectation, no-arbitrage

1. Introduction

This article focuses on the rationality of traditional financial models in predicting stock prices. By examining financial instruments such as bonds and shares of stock, this paper illustrates the intricate interplay between fixed interest rates, dividend streams, and stock prices. It explores the bond valuation under different capitalization methods and assesses the fundamental value of stocks through the lens of the No-Arbitrage Principle and adaptive expectations. The paper further delves into the concept of arbitrage and equilibrium in financial markets, demonstrating how these factors shape the stock price's evolution. By providing detailed mathematical models and solutions, including the derivation of key dynamic conditions and the steady-state price level, the model predicts the convergence of stock prices to their fundamental value over time with a minor assumption on stability of investors' expectation updating process.

2. Bond Valuation and Investment Returns

Bonds offer a predictable return at a fixed interest rate r . They are the cornerstone of the financial markets due to their low risk and stable returns. The price will, therefore, vary directly with the frequency of compounding of the interest, as that dictates the overall return that an investor gets on their investment in bonds. If the capitalization is annual, at time t , the value of a bond investment is:

$$V_B(t) = I * (1 + r)^{t-t_0}$$

where I = size of initial investment, r = annual interest rate and $(t-t_0)$ = time, in years, from when the investment was made. This equation now reflects the compounding nature of the interest rate: returns earn returns both on the initial investment and interest accrued over time.

The more frequent the compounding of interest, t is now monthly t , the bond investment value will shift as follows:

$$V_B(t) = I * \left(1 + \frac{r}{12}\right)^{12(t-t_0)}$$

The formula generalizes for n compounding periods per year:

$$V_B(t) = I * \left(1 + \frac{r}{n}\right)^{n(t-t_0)}$$

Finally, instantaneous or continuous compounding represents the limiting case where compounding occurs infinitely often, leading to the value of the bond being:

$$V_B(t) = I * \left(1 + \frac{r}{n}\right)^{n(t-t_0)}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{\frac{n}{r}} = e, \text{ as we can view } \frac{n}{r} = x,$$

$$\text{then } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\text{Therefore, } V_B(t) = I * e^{r(t-t_0)}$$

We can also find out the growth rate at t of the value of the investment made at t_0

$$V_B(t) = I * e^{rt} * e^{-rt_0}$$

$$\frac{dV_B(t)}{dt} = rIe^{rt} * e^{-rt_0}$$

$$= rI * e^{r(t-t_0)}$$

When $t = t_0$, the growth rate will be:

$$\frac{dV_B(t_0)}{dt} = rI * e^{r(t_0-t_0)} = rI$$

Similarly, this derivation employs the limit definition of the exponential function; that is, as the frequency of compounding goes to infinity, the bond approaches its limit. This continuous compounding finds great importance in theoretical models where a smooth, often tractable representation of the bond growth over time is needed.

3. Stock Valuation: Dividends and Capital Gains

Stocks differ from bonds because returns arrive in the form of dividends and possible capital gains. Stocks represent ownership in a firm and, hence, their performance is linked not only with the firm's performance, but also with the general conditions of the market. We assume that dividends, d , kept per unit of stock continuously arrive in, incorporating the income component in the stock returns. A stock investment I at any time t_0 involves the purchase $\frac{I}{P(t_0)}$ units of the stock at the prevailing market price $P(t_0)$ at time t_0 . In that respect, the value of an investment in stocks at any given time t is given as:

$$V_s(t) = \frac{I}{P(t_0)} [d(t - t_0) + P(t)]$$

Then, we can have the growth rate at t of the value of the investment made at t_0 :

$$\frac{dV_s(t)}{dt} = \frac{I}{P(t_0)} \left[d + \frac{dP(t)}{dt} \right]$$

We can also find out the growth rate at t_0 :

$$V_s(t_0) = \frac{I}{P(t_0)} [d(t - t_0) + P(t_0)] = \frac{I}{P(t_0)} * P(t_0) = I$$

$$\frac{dV_s(t)}{dt} = \frac{I}{P(t_0)} \left[d + \frac{dP(t)}{dt} \right] \text{ for } t = t_0$$

But we are not sure what the fundamental value of stock will be as time t approaches to infinity, we can take a integral to express our desired outcomes:

$$\begin{aligned} \int_{t_0}^{\infty} \frac{d}{e^{r(t-t_0)}} dt &= d \int_{t_0}^{\infty} e^{-r(t-t_0)} dt \\ &= e^{r(t_0)} * d * \int_{t_0}^{\infty} e^{-rt} dt \\ &= de^{rt_0} * \frac{e^{-rt}}{-r} \Big|_{t_0}^{\infty} \text{ spans from } t_0 \text{ to } \infty \\ &= de^{rt_0} \left(0 + \frac{1}{re^{rt_0}} \right) \\ &= \frac{d}{r} \end{aligned}$$

We can hence conclude that the fundamental value of the stock is constant over time. Here, I denote an initial investment and the first term in the above equation represents dividends accumulated in time, while the second one does the value of the stock at its current market price, $p(t)$. This simple bookkeeping underlines the two different sources of return from stocks: ongoing income from dividends and price appreciation. However, unlike bonds, stocks are riskier on grounds that their price fluctuations for instance, may get wild, since they are backed by company performance and investor's sentiment and macroeconomic conditions.

4. The Fundamental Value of Stocks

It defines the fundamental value of a stock as the present value of its expected dividend stream. Another critical concept in asset pricing, it will give the benchmark for determining whether this stock is overvalued or undervalued in light of its intrinsic worth. In this model, fundamental value is constant through time and given by the formula:

$$FV = \frac{d}{r}$$

For example, when a stock gives you a constant dividend over years

Year 1: $d = 100$; Year 2: $d = 100$; Year 3: $d = 100$...etc

We Assume r is not too big, so as time progresses, we will have:

$$= 100 + \frac{100}{1+r} + \frac{100}{(1+r)^2} + \dots + \frac{100}{(1+r)^n} + \dots = \frac{\left\{ 100 * \left(1 - \frac{1}{1+r} \right)^n \right\}}{\left\{ \left(1 - \frac{1}{1+r} \right) \right\}} = 100/r$$

This equation implies that the value of the stock varies directly with the dividend yield, and inversely with the interest rate on the bond. That is, the fundamental value of this stock rises with high dividends or low interest rates; the stock is attractive relative to bonds. Present value models, as

explored by Campbell and Shiller[1], demonstrate that stock prices should reflect the discounted stream of future dividends, a principle central to this paper's formulation. So we can know the dividends' role in determining stock intrinsic values and, through interest rates, investment decisions between bonds and stocks.

5. Arbitrage and Market Equilibrium

Arbitrage plays an essential role in the equilibration process of markets by exploiting price arbitrage opportunities between assets to attain risk-free profits. This model considers those opportunities provided by differences between expected returns from bonds and stocks; where those opportunities do exist, then the market must readjust to eliminate them, ensuring that in equilibrium, bonds and stocks at any one time yield the same expected return.

We model arbitrage in a simple framework with two assets: a risk-free asset, the bond, and a risky asset, the stock, whose returns are given in vector notation as:

- The risk-free asset: $a_f=[1,1]$, since it is earning a constant return in both market states.
- For the risky asset: $a_r=[2,-2]$, since for good market conditions the return is positive, while for bad conditions it is negative.

An investor can create a portfolio by buying z_f units of the risk-free asset and z_r units of the risky asset with a view to profiting from the differences in returns. The conditions for arbitrage are:

$$\begin{cases} z_f q_r + z_r q_f \leq 0 \\ z_f a_f + z_r a_r \geq 0 \\ z_f a_f + z_r a_r \neq 0 \end{cases}$$

These inequalities formulae assert that an investor pays an amount that is nonpositive at the outset in return for assuring returns that are nonnegative in every possible state of the market, thereby making a risk-free profit. For instance, when $q_r=-1.2$ and when $q_f=0.5$, the conditions allow for arbitrage. Lets consider this:

$$\begin{cases} 0.5z_f - 1.2z_r \leq 0 \\ z_f[1,1] + z_r[2,-2] \geq 0 \\ z_f[1,1] + z_r[2,-2] \neq 0 \end{cases}$$

$0.5z_f \leq 1.2z_r, \text{ which means } z_f \leq 2.4z_r$

$$\begin{cases} -z_f + 1.2z_r \geq 0 \\ z_f \geq 2z_r \end{cases}$$

$2z_r \leq z_f \leq 2.4z_r, \text{ hence, } z_f \in [2,2.4] \text{ if } z_r = 1$

However, when $q_f=1$, no arbitrage opportunity exists-apparently, it is a balancing act to maintain the equilibrium in financial markets.

Lets consider the above:

$$\begin{cases} 1z_f - 1.2z_r \leq 0 \\ z_f[1,1] + z_r[2,-2] \geq 0 \\ z_f[1,1] + z_r[2,-2] \neq 0 \end{cases}$$

$0.5z_f \leq 1.2z_r, \text{ which means } z_f \leq 2.4z_r$

$$\begin{cases} -z_f + 1.2z_r \geq 0 \\ z_f \geq 2z_r \end{cases}$$

$2z_r \leq z_f \leq 1.2z_r, \text{ hence, there is no solution}$

This means that no arbitrage possibilities should exist in general form, and expected returns from bonds and stocks should be equal in equilibrium. The no-arbitrage principle is a formal ground for financial theory because it ensures that markets work efficiently and there cannot exist persisting possibilities for risk-free profits[2]. The absence of arbitrage opportunities implies that the two assets (bonds and stocks) have the same expected return at any moment in time. The only uncertainty concerns the variation in the price of the stock. As the capital gain plus the dividend yield equals to the total return, so for a small part of time, bond and stock have the same expected return.

$$\begin{aligned} & \left\{ \begin{array}{l} \text{bond expected return: } I * r \\ \text{Stock: } \frac{I}{P(t_0)} \left[d + \frac{dP(t)}{dt} \right] \end{array} \right. \\ & \text{And } \frac{P(t_0 + \Delta x) - P(t_0)}{\Delta x} + \frac{d}{P(t_0)} = r \\ & P(t_0) * \frac{P(t_0 + \Delta x) - P(t_0)}{\Delta x} + d = r * P(t_0) \\ & P'(t_0) + d = rP(t_0) \end{aligned}$$

So, $P'(t_0) = rP(t_0) - d$

Economically, $P'(t_0)$ means the growth rate of the stock. Mathematically, $P'(t_0)$ means the rate of change of the stock market price at time t_0 . In order to calculate $P'(t_0)$ correctly, the current price $P(t_0)$ of the stock, the rate of return r , dividend d , and the specific t_0 are needed.

6. Adaptive Expectations and Price Dynamics

It implies that investors build up expectations concerning future stock prices based on past information, then gradually revise those forecasts as new information becomes available. This is what is referred to as adaptive expectations, and it basically plays a vital role in price dynamics due to the way it affects changes in stock prices over time. Chow provides empirical support for adaptive expectations, showing that investors systematically update their price forecasts based on historical data[3]. Such behavior can be captured by the model through the following differential equation:

$$p_e'(t) = \alpha[p(t) - p_e(t)]$$

where $p_e(t)$ is the expected price, $p(t)$ the actual price, and α an adjustment rate at which investors change their expectations. The term α indicates the adjustment speed-the higher the α , the quicker the response of the expectations towards the observed deviation in real price. Another way to get $p_e'(t)$ from our known conditions is that:

$$\text{known condition: } \begin{cases} p_e'(t) = \alpha[p(t) - p_e(t)] \\ P'(t_0) = rP(t_0) - d \end{cases}$$

$$\begin{aligned} p(t) &= \frac{p(t) + d}{r} \\ p_e'(t) &= \alpha \left[\frac{p_e'(t) + d}{r} - p_e(t) \right] \\ p_e'(t) &= \alpha \frac{p_e'(t)}{r} + \alpha \frac{d}{r} - \alpha p_e(t) \\ r p_e'(t) &= \alpha p_e'(t) + \alpha d - \alpha r p_e(t) \\ (r - \alpha) p_e'(t) &= \alpha d - \alpha r p_e(t) \\ p_e'(t) &= \frac{\alpha d}{r - \alpha} - \frac{\alpha r}{r - \alpha} p_e(t) \end{aligned}$$

In a steady state, denote $p_e = p_e^*$, we can derive another formula:

$$p_e'(t) = 0,$$

$$\begin{aligned} 0 &= \frac{\alpha d}{r - \alpha} - \frac{\alpha r}{r - \alpha} p_e^*(t) \\ \frac{\alpha d}{r - \alpha} &= \frac{\alpha r}{r - \alpha} p_e^*(t) \\ p_e^*(t) &= \frac{\alpha d}{r - \alpha} * \frac{r - \alpha}{\alpha r} \\ p_e^*(t) &= \frac{d}{r} \end{aligned}$$

This means that, at steady state, the expected price is the dividend over rate of return. Therefore, the expected price equals to the fundamental value at steady state.

Lets solve the below differential equation to get a general form of a solution:

$$\begin{aligned} p_e'(t) &= \frac{\alpha d}{r - \alpha} - \frac{\alpha r}{r - \alpha} p_e(t) \\ \frac{dp_e(t)}{dt} + \frac{\alpha r}{r - \alpha} p_e(t) &= \frac{\alpha d}{r - \alpha} \\ &\text{multiply } e^{\frac{\alpha r}{r - \alpha} t} \text{ on both side} \\ e^{\frac{\alpha r}{r - \alpha} t} * \frac{dp_e(t)}{dt} + e^{\frac{\alpha r}{r - \alpha} t} * \frac{\alpha r}{r - \alpha} p_e(t) &= e^{\frac{\alpha r}{r - \alpha} t} * \frac{\alpha d}{r - \alpha} \end{aligned}$$

By applying Leibniz Rule:

$$\begin{aligned} \frac{d(e^{\frac{\alpha r}{r - \alpha} t} * p_e(t))}{dt} &= \frac{\alpha r}{r - \alpha} * e^{\frac{\alpha r}{r - \alpha} t} \\ e^{\frac{\alpha r}{r - \alpha} t} * p_e(t) &= \int_0^t \frac{\alpha r}{r - \alpha} * e^{\frac{\alpha r}{r - \alpha} x} dx + C \\ e^{\frac{\alpha r}{r - \alpha} t} * p_e(t) &= \frac{\alpha r}{r - \alpha} \int_0^t e^{\frac{\alpha r}{r - \alpha} x} dx + C \\ e^{\frac{\alpha r}{r - \alpha} t} * p_e(t) &= \frac{\alpha r}{r - \alpha} * e^{\frac{\alpha r}{r - \alpha} t} * \frac{r - \alpha}{\alpha r} + C \\ p_e(t) &= \frac{d}{r} + C e^{\frac{\alpha r}{r - \alpha} t} \\ p_e(t_0) &= \frac{d}{r} + C e^{\frac{\alpha r}{r - \alpha} t_0}, \text{ and } C = p_e(t_0) e^{-\frac{\alpha r}{r - \alpha} t_0} - \frac{d}{r} e^{-\frac{\alpha r}{r - \alpha} t_0} \end{aligned}$$

Therefore,

$$p_e(t) = \frac{d}{r} + \left(p_e(t_0) - \frac{d}{r} \right) e^{-\frac{r\alpha}{r - \alpha}(t - t_0)}$$

This gives the solution of the expected price $p_e(t)$ converging over time to the fundamental value $\frac{d}{r}$. The rate of convergence depends here on adjustment speed α combined with the interest rate r , and this expression reflects how investor behavior influences market dynamics.

When time t approaches to infinity, $e^{-\frac{r\alpha}{r - \alpha}(t - t_0)}$ will approach to zero, so that our expected price will become $\frac{d}{r}$. Hence, as t approaches to infinity, our expected price will be the fundamental value. However, if $\alpha > r$ this means that our expected price is infinite, which is not realist. Therefore we need to assume $\alpha < r$, so that the investor's adjustment of expectation has an upperbound.

In conclusion, for $\alpha < r$, the expected price smoothly approaches the fundamental price. That stabilizes the market. When $\alpha > r$, the model predicts oscillatory or divergent behavior. Again, it underlines the desirability of bounded expectations for preventing market instability.

7. No-Arbitrage Condition and Stock Price Dynamics

Grossman and Stiglitz argue that perfect informational efficiency is impossible, which reinforces the need for adaptive expectations and gradual price adjustments as proposed in this model[4]. Moreover, the no-arbitrage condition provides the key interrelation among bond interest rate, stock dividend, and the growth rate of the stock price. As such, it furnishes a dynamic condition that has to hold in the market so that it may keep on being in equilibrium:

$$p'(t_0) = rp(t_0) - d$$

This equation says that the capital gains on the stock have to be such that the return from bonds is equated to the combined return that comes from both stock dividends and capital gains. Also by applying no-arbitrage condition previously stated, we can derive a new equation, the steps are listed below:

$$\begin{aligned} rp(t) - d &= \alpha[p(t) - p_e(t)] \\ (r - \alpha)p(t) &= d - \alpha p_e(t) \\ p(t) &= \frac{d - \alpha p_e(t)}{r - \alpha} \end{aligned}$$

In the previous calculation, we have that

$$p_e(t) = \frac{d}{r} + \left(p_e(t_0) - \frac{d}{r}\right) e^{-\frac{r\alpha}{r-\alpha}(t-t_0)}$$

Substituting the adaptive expectations model into the no-arbitrage condition yields the following law of motion for the stock price:

$$\begin{aligned} p(t) &= \frac{d}{r - \alpha} - \frac{\alpha}{r - \alpha} \left[\frac{d}{r} + \left(p_e(t_0) - \frac{d}{r}\right) e^{-\frac{r\alpha}{r-\alpha}(t-t_0)} \right] \\ p(t) &= \frac{dr - d\alpha}{r(r - \alpha)} - \frac{\alpha}{r - \alpha} \left(p_e(t_0) - \frac{d}{r}\right) e^{-\frac{r\alpha}{r-\alpha}(t-t_0)} \\ p(t) &= \frac{d}{r} - \frac{\alpha}{r - \alpha} \left(p_e(t_0) - \frac{d}{r}\right) e^{-\frac{r\alpha}{r-\alpha}(t-t_0)} \end{aligned}$$

This formulation shows, in particular, how no-arbitrage condition is interwoven with adaptive expectations in the process in which stock prices tend toward their fundamental values over a period of time as $e^{-\frac{r\alpha}{r-\alpha}(t-t_0)}$ approaches to 0 when time approaches to infinity. The condition $\alpha < r$ is crucial because it means that the adjustment process be productive of a stable equilibrium. For $\alpha > r$, adjustment becomes too aggressive and leads to unrealistic price behavior like bubbles or crashes.

8. Conclusion

This study introduces a theoretical framework for understanding how stock prices evolve by combining ideas from bond and stock valuation, arbitrage, and adaptive expectations. Using differential equations, the research shows that under the no-arbitrage condition, stock prices naturally move toward their fundamental values. The study also highlights the importance of keeping investor expectations in check—when they react too strongly, it can lead to market instability and extreme price movements, like bubbles or crashes. Lo's Adaptive Markets Hypothesis supports the idea that markets evolve in response to investor learning and environmental changes, aligning with the adaptive expectations model proposed in this study[5]. By examining how bond returns, stock dividends, and investor behavior interact, the research sheds light on the key forces driving market adjustments. Ultimately, this model emphasizes the role of rational expectations and adaptive processes in maintaining balance in financial markets, which are devote to be helpful for both financial theory and real-world investment practices.

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