

Theoretical Analysis of Dependence Structures for Credit Risk+ Framework

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Abstract: The Credit Risk+ model is an influential method to evaluate the default risk level of a credit portfolio in the financial and insurance industry. As an essential modelling technique in the Credit Risk+ framework, dependence structure models relationship between default probability and risk factors using a variety of statistical methods. This research paper analyzes the dependence structure related to default correlation and sector dependence among classic and enhanced Credit Risk+ models. At first, the concepts of default event and default probability are discussed within classic models for explaining the connection between default correlation and risk factors. Then, given the limitations of classic models, two enhanced models are presented to incorporate sector dependence generated by additional Gamma-distributed variables. Based on previous literature and probability theory, this paper revisits methods of deriving the correlation coefficient for default dependence and the variance-covariance for sector dependence. The proof results show that default correlation is associated with default probability and sector characteristics. Also, the multi-system model, which develops sector dependence from a set of independent macro-economic variables via linear combination, provides a more flexible variance-covariance structure than the compound Gamma model.

Keywords: Credit Risk+, Default probability, Default correlation, Sector dependence, Gamma distribution.

1. Introduction

Credit Risk+ (CR+) was developed by CREDIT Suisse Financial Products (CSFP) in October 1997 as its portfolio model for credit risk management. The model has quickly gained popularity among financial practitioners and industries because CR+ applied actuarial mathematics to analyze default events and measure default portability, which is the core concept in credit risk analysis [1].

Afterwards, the theoretical extension of CR+ model focuses on building a flexible dependence structure among risk factors, and designing an efficient algorithm for loss distribution approximation and risk measure calculation. For example, Wang et al. proposed a conditional compound Poisson model to approximate loss distribution by modelling correlation among common risk factors as a class of extreme copulas [2]. Also, several algorithms, such as Johnson curve fitting, saddle-point approximation, importance sampling, and check function formulation, were demonstrated with efficiency to compute risk measures in CR+ frameworks by Huang and Kwok [3]. However, the current literature lacks a combination of theoretical analysis and practical techniques. So far, only

Bolder's literature has shown how Python code is implemented for CR+ modelling components via Monte-Carlo methods and NumPy tools [4].

This study aims to investigate the dependence structure across default probabilities and risk factors for multiple CR+ models. Firstly, this paper shows how default correlation and sector dependence arise due to different model assumptions when the CR+ model is extended from a simple framework to several enhanced frameworks. Secondly, based on probability theory, the mathematical expression of default correlation and sector dependence will be derived for the specified CR+ model with a correlation coefficient and variance-covariance (VCV) structure. In addition, dependence structures and derived formulas are compared between different CR+ models. Because default correlation and sector dependence are key elements for determining portfolio credit risk, this paper will bridge the gap between introductory statistical methodologies and complex risk modelling techniques in CR+ academic work.

2. Dependence Structure

2.1. Default Event

For credit risk modelling, the default event for obligor n is defined by an indicator $\mathbb{I}$ with a binary outcome, which arises if the number of default occurrences, denoted as a random variable X_n , is greater than or equal to one as following:

$$\mathbb{I}_{\{X_n \geq 1\}} = \begin{cases} 0: \text{non - default} \\ 1: \text{default} \end{cases} \quad (1)$$

For example, if a default occurs one or more times before payment due date, the default indicator equals to one. Otherwise, it equals to zero, which indicates a trustworthy obligor without default occurrence.

2.2. Default Probability

Starting from a simple model that assumes independence between individual default event, X_n is defined as an independent and identically Poisson distributed random variable for all obligors $n = 1, 2, \dots, N$. Thus, the probability of a default event with X_n can be approximated by the probability mass function of the Poisson distribution and driven by the Poisson arrival intensity parameter λ_n (mean number of default occurrences) as follows:

$$X_n \sim \text{i. d. Poisson}(\lambda_n); \mathbb{P}(X_n = k) = \frac{e^{-\lambda_n} \lambda_n^k}{k!}, k \in \mathbb{N}_+ \quad (2)$$

Wherein, the probability of default event $\mathbb{P}(X_n = k)$ in simple default model is also considered as unconditional default probability for obligor n in the CR+ model framework [4].

Next, in the original version of the CR+ model developed by Tom Wilde of CSFP, the default probability of obligor n is extended to a stochastic random variable $p_n(S)$, which depends on a set of risk factors $S = (S_1, S_2, \dots, S_K)$ via a linear relationship:

$$p_n(S) = p_n(\omega_{n,0} + \omega_{n,1}S_1 + \omega_{n,2}S_2 + \dots + \omega_{n,K}S_K) \quad (3)$$

p_n is the unconditional default probability of obligor n , and $\omega_{n,k}$ is the risk factor loading. Given the dependence on common risk factors, $p_n(S)$ is considered as conditional default probability of obligor n .

$$p_n(S) = \mathbb{P}(\mathbb{I}_{\{X_n \geq 1\}} = 1 | S) \quad (4)$$

In addition, to ensure that expectation of $p_n(S)$ equals to the unconditional default probability p_n , a constraint of summing each sector loading $\omega_{n,k}$ to unity is imposed by:

$$\sum_{k=0}^K \omega_{n,k} = 1, k = 0, 1, 2, \dots, K \quad (5)$$

According to Avesani et al., risk factor S_k incurs default correlation and is viewed as latent variables associated with a “sector” that represents the industrial sector, or demographic region [5]. Therefore, individual default events is now assumed as mutually independent only conditional on sectors [5].

2.3. Default Correlation

For CR+ frameworks, correlation between any two individual default events X_n, X_m can be measured by calculating the correlation coefficient ρ for two default indicator $\mathbb{I}_{\{X_n \geq 1\}}$ and $\mathbb{I}_{\{X_m \geq 1\}}$ as follows:

$$\rho(\mathbb{I}_{\{X_n \geq 1\}}, \mathbb{I}_{\{X_m \geq 1\}}) = \frac{\text{Cov}(\mathbb{I}_{\{X_n \geq 1\}}, \mathbb{I}_{\{X_m \geq 1\}})}{\sqrt{\text{var}(\mathbb{I}_{\{X_n \geq 1\}})} \sqrt{\text{var}(\mathbb{I}_{\{X_m \geq 1\}})}} \quad (6)$$

Then, using the laws of iterated expectation and conditional independence, Bolder simplified the variance-covariance structure of the default indicator as follows [4]:

$$\mathbb{E}(\mathbb{I}_{\{X_n \geq 1\}}) = p_n; \text{Var}(\mathbb{I}_{\{X_n \geq 1\}}) = p_n(1 - p_n) \quad (7)$$

$$\begin{aligned} \text{Cov}(\mathbb{I}_{\{X_n \geq 1\}}, \mathbb{I}_{\{X_m \geq 1\}}) &= \mathbb{E} \left[\left[\mathbb{I}_{\{X_n \geq 1\}} - \mathbb{E}(\mathbb{I}_{\{X_n \geq 1\}}) \right] \left[\mathbb{I}_{\{X_m \geq 1\}} - \mathbb{E}(\mathbb{I}_{\{X_m \geq 1\}}) \right] \right] \\ &= \mathbb{E}(\mathbb{I}_{\{X_n \geq 1\}} \mathbb{I}_{\{X_m \geq 1\}}) - p_n p_m \\ &= \mathbb{E}[\mathbb{E}(\mathbb{I}_{\{X_n \geq 1\}} \mathbb{I}_{\{X_m \geq 1\}} | S)] - p_n p_m \\ &= \mathbb{E}[p_n (\sum_{k=0}^K \omega_{n,k} S_k) \cdot p_m (\sum_{k=0}^K \omega_{m,k} S_k)] - p_n p_m \\ &= p_n p_m [\sum_{k=0}^K \omega_{n,k} \omega_{m,k} \mathbb{E}(S_k^2) - \sum_{k=0}^K \omega_{n,k} \omega_{m,k}] \end{aligned} \quad (8)$$

Following the original CR+ model, because S_k is also assumed to be mutually independent and Gamma distributed with shape parameter α_k and scale parameter β_k , that satisfies:

$$\mathbb{E}(S_k) = \alpha_k \beta_k = 1; \text{Var}(S_k) = \alpha_k \beta_k^2 = \sigma_k^2 \quad (9)$$

the second moment of S_k can be simplified and substituted into equation (9) to yield the correlation coefficient as follows:

$$\mathbb{E}(S_k^2) = \text{Var}(S_k) + \mathbb{E}(S_k)^2 = \sigma_k^2 + 1 \quad (10)$$

$$\text{Cov}(\mathbb{I}_{\{X_n \geq 1\}}, \mathbb{I}_{\{X_m \geq 1\}}) = p_n p_m (\sum_{k=1}^K \sigma_k^2 \omega_{n,k} \omega_{m,k}) \quad (11)$$

$$\rho(\mathbb{I}_{\{X_n \geq 1\}}, \mathbb{I}_{\{X_m \geq 1\}}) = (\sum_{k=1}^K \sigma_k^2 \omega_{n,k} \omega_{m,k}) \left(\frac{p_n p_m}{\sqrt{p_n(1-p_n)} \sqrt{p_m(1-p_m)}} \right) \quad (12)$$

2.4. Sector Dependence

2.4.1. Uniform covariance

Although the original CR+ model can generate default correlation among obligors, it neglects the inter-dependency among sectors. Thus, Giese extended the original CR+ model to the compound

Gamma model in 2004 by introducing an additional Gamma variable γ , which drives correlated changes between sectors:

$$S_k \sim \text{Gamma}(\alpha_k, \beta_k) \quad (13)$$

$$\alpha_k = \gamma \bar{\alpha}_k \quad (14)$$

where α_k is a function of γ , β_k and $\bar{\alpha}_k$ are constant [5].

According to Avesani et al., sectors are now assumed as mutually independent only conditional on Gamma variable γ which satisfies [5]:

$$\mathbb{E}(\gamma) = 1; \text{Var}(\gamma) = \sigma^2 \quad (15)$$

Hence, sector dependence can be assessed by exploring VCV structure of sectors S_k with consideration of Gamma variable γ . At first, conditional expectation of S_k on γ are denoted as:

$$\mathbb{E}(S_k|\gamma) = \alpha_k \beta_k = \gamma \bar{\alpha}_k \beta_k \quad (16)$$

Then, based on the assumption that unconditional expectation of S_k equals to 1, $\beta_k = \frac{1}{\bar{\alpha}_k}$ can be obtained using the law of iterated expectation and further applied to deriving the conditional variance of S_k as follows [5]:

$$\mathbb{E}(S_k) = \mathbb{E}[\mathbb{E}(S_k|\gamma)] = \bar{\alpha}_k \beta_k \mathbb{E}(\gamma) = \bar{\alpha}_k \beta_k = 1 \quad (17)$$

$$\text{Var}(S_k|\gamma) = \alpha_k \beta_k^2 = \frac{\gamma}{\bar{\alpha}_k} \quad (18)$$

Next, using the alternative form of marginal variance, unconditional variance of S_k is given by:

$$\text{Var}(S_k) = \mathbb{E}[\text{Var}(S_k|\gamma)] + \text{Var}[\mathbb{E}(S_k|\gamma)]$$

$$= \mathbb{E}\left(\frac{\gamma}{\bar{\alpha}_k}\right) + \text{Var}(\gamma \bar{\alpha}_k \beta_k)$$

$$= \beta_k + (\bar{\alpha}_k \beta_k)^2 \text{Var}(\gamma)$$

$$= \beta_k + \sigma^2 \quad (19)$$

Finally, using product expression of covariance and the dependence structure of sectors, the unconditional covariance between any two sectors S_n and S_m is resulted by a constant that equals to the variance of γ as follows:

$$\begin{aligned} \text{Cov}(S_n, S_m) &= \mathbb{E}(S_n \cdot S_m) - \mathbb{E}(S_n) \cdot \mathbb{E}(S_m) \\ &= \mathbb{E}[\mathbb{E}(S_n \cdot S_m|\gamma)] - \bar{\alpha}_n \beta_n \bar{\alpha}_m \beta_m \\ &= \mathbb{E}[\mathbb{E}(S_n|\gamma) \cdot \mathbb{E}(S_m|\gamma)] - \bar{\alpha}_n \beta_n \bar{\alpha}_m \beta_m \\ &= \bar{\alpha}_n \beta_n \bar{\alpha}_m \beta_m [\mathbb{E}[\gamma^2] - 1] \\ &= [\text{Var}(\gamma) + \mathbb{E}(\gamma)^2 - 1] \\ &= \sigma^2 \end{aligned} \quad (20)$$

2.4.2. Non-uniform covariance

However, a uniform covariance structure between sectors is obviously unrealistic. Subsequently, to rebuild a flexible covariance structure, the compound Gamma model was further extended by

modeling sector dependence through connecting S_k with a set of independent macro-economic variables, such as the multi-system model developed by Peng and Lv [6].

In the multi-system model, the shape parameter of S_k is a linear combination of independent macro-economic variables Y_i that influence sector correlation with the following set-ups:

$$\alpha_k = (c_{k1}Y_1 + c_{k2}Y_2 + \dots + c_{kL}Y_L)\bar{\alpha}_k \quad (21)$$

$$\sum_{l=1}^L c_{kl} = 1, l = 1, 2, \dots, L \quad (22)$$

where $\bar{\alpha}_k$ is a constant.

According to Peng and Lv, sectors are assumed as mutually independent only conditional on macro-economic variables Y_i 's which satisfies [6]:

$$\mathbb{E}(Y_l) = 1; \text{Var}(Y_l) = \delta_l^2 \quad (23)$$

So, similar to the VCV simplification steps in section 2.4.1, the rest part will adjust Peng and Lv's proof to study VCV structure of S_k . At first, the conditional expectation of S_k is given by:

$$\mathbb{E}(S_K|Y_1, Y_2, \dots, Y_N) = \alpha_k \beta_k = (c_{k1}Y_1 + c_{k2}Y_2 + \dots + c_{kN}Y_N)\bar{\alpha}_k \beta_k \quad (24)$$

Then, the unconditional expectation of S_k is given by:

$$\begin{aligned} \mathbb{E}(S_K) &= \mathbb{E}(\mathbb{E}(S_K|Y_1, Y_2, \dots, Y_N)) \\ &= \bar{\alpha}_k \beta_k [c_{k1}\mathbb{E}(Y_1) + c_{k2}\mathbb{E}(Y_2) + \dots + c_{kL}\mathbb{E}(Y_L)] \\ &= \bar{\alpha}_k \beta_k = 1 \end{aligned} \quad (25)$$

Again, substituting $\beta_k = \frac{1}{\bar{\alpha}_k}$ into the conditional variance of S_k as follows:

$$\begin{aligned} \text{Var}(S_K|Y_1, Y_2, \dots, Y_N) &= \alpha_k \beta_k^2 \\ &= \bar{\alpha}_k (c_{k1}Y_1 + c_{k2}Y_2 + \dots + c_{kN}Y_N) \beta_k^2 \\ &= \frac{(c_{k1}Y_1 + c_{k2}Y_2 + \dots + c_{kN}Y_N)}{\bar{\alpha}_k} \end{aligned} \quad (26)$$

Next, the unconditional variance of S_k can be derived as:

$$\begin{aligned} \text{Var}(S_K) &= \mathbb{E}[\text{Var}(S_K|Y_1, Y_2, \dots, Y_L)] + \text{Var}[\mathbb{E}(S_K|Y_1, Y_2, \dots, Y_L)] \\ &= \mathbb{E}\left[\frac{(c_{k1}Y_1 + c_{k2}Y_2 + \dots + c_{kL}Y_L)^2}{\bar{\alpha}_k^2}\right] + \text{Var}[\alpha_k \beta_k] \\ &= \beta_k + (\bar{\alpha}_k \beta_k)^2 \text{Var}[c_{k1}Y_1 + c_{k2}Y_2 + \dots + c_{kL}Y_L] \\ &= \beta_k + \sum_{l=1}^L c_{kl}^2 \delta_l^2 \end{aligned} \quad (27)$$

At the end, the most challenging part is to obtain the unconditional covariance between two sectors where detailed simplification is shown as below:

$$\begin{aligned} \text{Cov}(S_K, S_l) &= \mathbb{E}[\mathbb{E}(S_n|Y_1, \dots, Y_L) \cdot \mathbb{E}(S_m|Y_1, \dots, Y_L)] - \bar{\alpha}_n \beta_n \bar{\alpha}_m \beta_m \\ &= \bar{\alpha}_n \beta_n \bar{\alpha}_m \beta_m [\mathbb{E}((c_{n1}Y_1 + \dots + c_{nL}Y_L)(c_{m1}Y_1 + \dots + c_{mL}Y_L)) - 1] \\ &= (\sum_{l=1}^L c_{nl}c_{ml} \mathbb{E}[Y_l^2]) + (\sum_{l=1}^L c_{nl}(1 - c_{ml}))\mathbb{E}(Y_l)^2 - 1 \\ &= (\sum_{l=1}^L c_{nl}c_{ml}(\text{Var}(Y_l) + \mathbb{E}(Y_l)^2)) - (\sum_{l=1}^L c_{nl}c_{ml}) \\ &= \left(\sum_{l=1}^L c_{nl}c_{ml}\right)(\text{Var}(Y_l) + \mathbb{E}(Y_l)^2 - 1) \\ &= \sum_{l=1}^L c_{nl}c_{ml}\delta_l^2 \end{aligned} \quad (28)$$

Table 1: (continued).

3. Model Comparison

Comparison of dependence structure for different CR+ frameworks, including independent default model (M1), original CR+ model (M2), compound Gamma model (M3), and multi-system model (M4) will be broken down into four categories, model assumption, parameters, default correlation, and sector dependence. At first, Table 1 illustrates what factor introduces dependency and how such dependency changes each model's assumptions. It is found that the assumption of default independence in M1 is expanded to conditional independence with sectors assigned in M2. Likely, the assumption of sector independence in M2 is expanded to conditional independence with additional factor or economic variables assigned to M3 and M4.

Table 1: Comparison of Assumptions between Different CR+ Models

	Assumptions	Dependence Structures
M1	default events are independent between obligors	NA
M2	• default events are mutually independent conditional on sectors S_k • sectors are independent	sectors S_k introduces default correlation
M3	• default events are mutually independent conditional on sectors S_k • sectors are mutually independent conditional on factor γ	• sectors S_k introduces default correlation • factor γ introduces sector dependence
M4	• default events are mutually independent conditional on sectors S_k • sectors are mutually independent conditional on economic variables Y_i • economic variables are independent	• sectors S_k introduces default correlation • economic variables Y_i introduces sector dependence

Next, Table 2 demonstrates how statistical features of parameters underpinning dependence structure varies between different CR+ models. Both sector parameters S_k and other parameters (single factor γ and economic variables Y_1) follow Gamma distribution with unconditional mean equal to one. Default correlation stems from incorporating sector parameters into conditional default probability formula with sector weights via linear relationship. While, sector dependence is attributed to build shape parameter of sector as a linear function of single factor γ or economic variables Y_1 . Moreover, in M3 and M4, either mean or variance of sector is assumed to be distinguished between conditional value and unconditional value.

Table 2: Comparison of Parameters between Different CR+ Models

	Sector Parameters		Other Parameters
M1	NA		NA
M2	• $p_n(S) = p_n(\omega_{n,0} + \omega_{n,1}S_1 + \dots + \omega_{n,k}S_k)$	• $\mathbb{E}(S_K) = 1$ • $\text{Var}(S_K) = \sigma_k^2$	NA

M3	• $\sum_{k=0}^K \omega_{n,k} = 1$ • $S_k \sim \text{Gamma}(\alpha_k, \beta_k)$	• $\mathbb{E}(S_K) = 1$ • $\mathbb{E}(S_K \gamma) = \alpha_k \beta_k$ • $\text{Var}(S_K \gamma) = \alpha_k \beta_k^2$	• $\alpha_k = \gamma \bar{\alpha}_k$ • γ is Gamma r.v. • $\mathbb{E}(\gamma) = 1; \text{Var}(\gamma) = \sigma^2$
M4		• $\mathbb{E}(S_K) = 1$ • $\mathbb{E}(S_K Y_1, \dots, Y_L) = \alpha_k \beta_k$ • $\text{Var}(S_K Y_1, \dots, Y_L) = \alpha_k \beta_k^2$	• $\alpha_k = (c_1 Y_1 + \dots + c_L Y_L) \bar{\alpha}_k$ • $\sum_{l=1}^L c_{kl} = 1, l = 1, \dots, L$ • Y_i is Gamma r.v. • $\mathbb{E}(Y_l) = 1; \text{Var}(Y_l) = \delta_l^2$

In terms of default correlation, Table 3 summarizes closed-form formula of correlation coefficient for each CR+ model. According to equation (6), each CR+ model shares a fixed numerator structure of correlation coefficient which only contains unconditional default probability for two obligors. If M3 and M4 are treated as enhanced CR+ models, their denominator of correlation coefficient are more complex than the original CR+ model (M1) because they yield complex structure of second moment $\mathbb{E}(S_K^2)$ in equation (8) that consists of parameters related to sectors and other factor variables.

Table 3: Comparison of Default Correlation between Different CR+ Models

	Correlation Coefficient
M1	NA
M2	$\rho = \left(\sum_{k=1}^K \sigma_k^2 \omega_{n,k} \omega_{m,k} \right) \left(\frac{p_n p_m}{\sqrt{p_n(1-p_n)} \sqrt{p_m(1-p_m)}} \right)$
M3	$\rho = \left[\sum_{k=1}^K (\beta_k + \sigma^2) \omega_{n,k} \omega_{m,k} \right] \left(\frac{p_n p_m}{\sqrt{p_n(1-p_n)} \sqrt{p_m(1-p_m)}} \right)$
M4	$\rho = \left[\sum_{k=1}^K \left(\beta_k + \sum_{l=1}^L c_{kl}^2 \delta_l^2 \right) \omega_{n,k} \omega_{m,k} \right] \left(\frac{p_n p_m}{\sqrt{p_n(1-p_n)} \sqrt{p_m(1-p_m)}} \right)$

For comparison of sector dependence in Table 4, M4 is observed with complex outputs for unconditional VCV of sectors that include scale parameters of sector, and each macro-economic variable's weight and variance. A particular case is that M3 produces an uniform covariance that only equals to the variance of factor γ .

Table 4: Comparison of Sector Dependence between Different CR+ Models

	Unconditional Variance of Sector	Unconditional Covariance between Sector
M1	NA	NA
M2	$\text{Var}(S_K) = \sigma_k^2$	NA
M3	$\text{Var}(S_K) = \beta_k + \sigma^2$	$\text{Cov}(S_n, S_m) = \sigma^2$

M4	$\text{Var}(S_K) = \beta_k + \sum_{l=1}^L c_l^2 \delta_l^2$	$\text{Cov}(S_n, S_m) = \sum_{l=1}^L c_{nl} c_{ml} \delta_l^2$
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4. Conclusion

This paper has analyzed how dependence structures, which involves the relationship between default probability and risk factors, are incorporated into different CR+ modelling frameworks. Based on several academic studies, this paper also revisits mathematical proof for measuring default correlation and sector dependence by using conditional expectation to simplify some challenging steps of the variance-covariance structure. After comparing dependence structures in different CR+ models, four conclusions are arrived at:

Firstly, compared to unconditional probability approximated by the Poisson distribution from the independent-default model, the CR+ frameworks introduced conditional default probability as a stochastic random variable that is dependent on idiosyncratic and systematic risk factors, and thus generates correlation between individual default events.

Secondly, extension from original CR+ model to enhanced CR+ models is to incorporate sector dependence through modelling sector parameters with other variables. For example, the compound Gamma model sets the shape parameter of sector as a single Gamma distributed variable. Multi-system model assumes the shape parameter of sector dependent on macro-economic factors via linear relationship.

Thirdly, derived formulas of correlation coefficient ρ show that weight and second moment of each sector, plus each obligor's unconditional default probability contribute to default correlation. While, enhanced CR+ models produce more complicated ρ structure due to sector dependence.

Fourthly, complex VCV structures of sectors are also derived for enhanced CR+ models with the exception of uniform covariance in compound Gamma model. The unconditional covariance formula for multi-system model manifests that the sector dependence is fully driven by each macro-economic variable's weight and variance.

For limitation, it is necessary to test significance of the correlation between different CR+ models based on numerical analysis. Therefore, future research is suggested to explore whether the default correlation in enhanced CR+ models is significantly stronger than the original CR+ model. Or, what kind of VCV structure between sectors is associated with a higher level of sector dependence.

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